A PROPERTY OF MULTINOMIAL COEFFICIENTS

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ABSTRACT

The multinomial coefficients "surrounding" a given multinomial coefficient in a generalized Pascal pyramid are partitioned into subsets such that the product of the coefficients in each subset is a constant N and such that the product of all the coefficients "surrounding" a given m-nomial coefficient is N^m. The result is then generalized to other numerical triangles or pyramids.

1. INTRODUCTION

In the paper by Hansell and Hoggatt [1] the following is proved:

Theorem. The product of the six binomial coefficients surrounding each binomial coefficient $\binom{n}{k}$, $(n \geq 2;\ 0 < k < n)$, in Pascal's triangle is a perfect integer square, N^2. Further, each triad formed by taking alternate binomial coefficients has product N.

Further results in the plane are obtained by Gould in [4].

In this paper, we generalize this theorem to generalized Pascal pyramids in m-space.

2. SELECTING THE MULTINOMIAL COEFFICIENTS

Let us expand $(x_1 + x_2 + x_3 + \cdots + x_m)^n$, $(m \geq 2;\ n = 0, 1, 2, \cdots)$:

$$(x_1 + \cdots + x_m)^n = \sum_{\substack{k_1+\cdots+k_m=n \\ (k_j \geq 0)}} \binom{k_1 + k_2 + \cdots + k_m}{k_1,\ k_2,\ \cdots\ k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} .$$

Here

$$\binom{k_1 + k_2 + \cdots + k_m}{k_1, \quad k_2, \quad \cdots, \quad k_m} = \frac{(k_1 + k_2 + \cdots + k_m)!}{k_1!\,k_2!\,k_3!\,\cdots\,k_m!} \quad .$$

The recurrence relation is

$$\text{(R)} \quad \binom{k_1 + k_2 + \cdots + k_m}{k_1, \quad k_2, \quad \cdots, \quad k_m} = \sum_{j=1}^{m} \binom{k_1 + k_2 + \cdots + k_m - 1}{k_1 - \delta_{1j},\ k_2 - \delta_{2j},\ \cdots,\ k_m - \delta_{mj}}.$$

where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$.

Given a multinomial coefficient (inside the pyramid, i.e., $k_s \geq 1$, $s = 1, 2, \cdots, m$)

$$A = \binom{k_1 + k_2 + \cdots + k_m}{k_1, \quad k_2, \quad \cdots, \quad k_m}$$

there are m multinomial coefficients

$$\binom{k_1 + k_2 + k_3 + \cdots + k_m - 1}{k_1 - \delta_{1j},\ k_2 - \delta_{2j},\ \cdots,\ k_m - \delta_{mj}}, \qquad (j = 1, 2, \cdots, m)$$

which "contribute to A" by means of recurrence relation (R); that is, lie directly above A in the pyramid. These same m multinomial coefficients contribute to $m(m - 1)$ multinomial coefficients

$$\binom{k_1 + k_2 + \cdots + k_{m-1} + k_m}{k_1 - \delta_{1j} + \delta_{1k},\ k_2 - \delta_{2j} + \delta_{2k},\ \cdots,\ k_m - \delta_{mj} + \delta_{mk}}, \quad (j, k = 1, 2, \cdots, m; j \neq k)$$

which are all on the same level as A. There are also m multinomial coefficients which are contributed to by A, namely those of the form

$$\binom{k_1 + k_2 + \cdots + k_m + 1}{k_1 + \delta_{1j},\ k_2 + \delta_{2j},\ \cdots,\ k_m + \delta_{mj}}, \quad (j = 1, 2, \cdots, m) \ .$$

Thus there are m above A, $m(m-1)$ on the same level as A and m below A. These $m(m+1)$ multinomial coefficients we say are adjacent to A, and geometrically surround A.

3. THE PRINCIPAL RESULT

Theorem. The product of the $m(m+1)$ multinomial coefficients adjacent to A is a perfect integer m^{th} power.

Proof. In the following, $s = 1, 2, \cdots, m$. On the level above A, the number $k_s - 1$ appears once; on the level with A the number $k_s - 1$ appears $m - 1$ times; $k_s - 1$ does not appear in the level below A. (In the level with A, k_s appears $(m-1)(m-2)$ times; and on the level below A, k_s appears $m - 1$ times. On the level above A, $k_s + 1$ does not appear; on the level with A, $k_s + 1$ appears $(m - 1)$ times; and on the level below A, $k_s + 1$ appears once. Thus, in the denominator of the product, $(k_s - 1)!$ appears m times, $(k_s)!$ appears $m(m-1)$ times, and $(k_s + 1)!$ appears m times. The product, therefore, of all $m(m+1)$ multinomial coefficients adjacent to A is:

$$P = \frac{\left[\left(\sum_{i=1}^{m} k_i - 1\right)!\right]^m \left[\left(\sum_{i=1}^{m} k_i\right)!\right]^{m(m-1)} \left[\left(\sum_{i=1}^{m} k_i + 1\right)!\right]^m}{\prod_{i=1}^{m} [k_i - 1!]^m [k_i!]^{m(m-1)} [(k_i + 1)!]^m}$$

$$= \left[\frac{\left(\sum_{i=1}^{m} k_i - 1\right)! \left[\left(\sum_{i=1}^{m} k_i\right)!\right]^{m-1} \left(\sum_{i=1}^{m} k_i + 1\right)!}{\prod_{i=1}^{m} (k_i - 1)!\,(k_i!)^{m-1}(k_i + 1)!}\right]^m = N^m$$

N. B. N is an integer, since (p/q) reduced to lowest terms with $q \neq 1$ is not an integer when raised to the m^{th} power. But the product is an integer, since each multinomial coefficient factor is an integer.

We next prove the following rather surprising result.

Theorem: The $m(m+1)$ multinomial coefficients adjacent to A with product N^m, can be decomposed into m sets of $(m+1)$ multinomial coefficients such that the product over each set is N. Furthermore, the construction yields sets of $(m+1)$ multinomial coefficients such that permuting the subscripts cyclically on any one set $m-1$ times produces all the other sets. Thus the m sets are congruent by rotation.

Proof. We now describe a construction for the sets. Recall that the product within each set must be

$$N = \frac{\left(\sum_{i=1}^{m} k_i - 1\right)!\left(\sum_{i=1}^{m} k_i\right)!^{m-1}\left(\sum_{i=1}^{m} k_i + 1\right)!}{\prod_{i=1}^{m} (k_i - 1)!(k_i!)^{m-1}(k_i + 1)!}$$

For convenience, we introduce the following notation for the multinomial coefficient

$$\binom{k_1 + k_2 + \cdots + k_m}{k_1, \ k_2, \ \cdots, \ k_m} = (0, 0, 0, \cdots, 0)$$

so that by introducing -1 or $+1$ as entries in the m-tuple, we can raise or lower one of the k_i and thus represent adjacent coefficients. For example:

$$\binom{k_1 + k_2 + \cdots + k_m - 1}{k_1 - 1, \ k_2, k_3, \cdots, k_m} = (-1, 0, 0, \cdots, 0)$$

and

$$\binom{k_1 + k_2 + \cdots + k_m}{k_1, k_2, k_3 + 1, k_4, \cdots, k_{m-1}, k_m - 1} = (0, 0, 1, 0, \cdots, 0, -1) \quad .$$

Thus a subset of multinomial coefficients of the type desired could be represented as an $(m+1) \times m$ matrix, where each row is a vector as described above, each row representing one of the adjacent multinomial coefficients. Each subset, in order to have the proper numerator in the product, must have one coefficient from above, one from below, and $m - 1$ from the same level as the given coefficient. We shall adopt the convention that the first row represents the coefficient above and the $(m + 1)^{st}$ row the coefficient below. Let the $(m + 1) \times m$ matrix have entries a_{ij}. It is necessary to consider two separate cases.

For m odd, let

$$\left.\begin{aligned} a_{jj} &= -1 \\ a_{m+2-j,j} &= +1 \\ a_{ij} &= 0 \text{ otherwise} \end{aligned}\right\} \begin{aligned} i &= 1, 2, \cdots, m + 1 \\ j &= 1, 2, \cdots, m \end{aligned} \quad .$$

We illustrate with $m = 5$:

$$C = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & +1 \\ 0 & 0 & -1 & +1 & 0 \\ 0 & 0 & +1 & -1 & 0 \\ 0 & +1 & 0 & 0 & -1 \\ +1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \\ 3 \\ 1 \\ 4 \\ 2 \\ \\ \end{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \begin{matrix} \text{Spacing between } -1 \text{ and } +1 \\ \text{on the middle } m - 1 \text{ rows .} \end{matrix}$$

This corresponds to six multinomial coefficients whose 5 lower arguments are given in rows of this matrix. The other four sets are obtained by rotating cyclically the column vectors of matrix C. We note that $(k_s - 1)!$ appears once, $k_s!$ appears $m - 1$ times, and $(k_s + 1)!$ appears once in each of the five sets, $s = 1, 2, \cdots, m$.

For m even, let

$$\begin{aligned} a_{jj} &= -1 && j = 1, 2, \cdots, m \\ a_{k+1,m+1-k} &= +1 && k = 1, 2, \cdots, (m/2) - 1 \\ a_{m+1-k,k} &= +1 && k = 1, 2, \cdots, (m/2) + 1 \\ a_{m+1,(m/2)+1} &= 1 \\ a_{ij} &= 0 \text{ otherwise} && \left\{\begin{aligned} i &= 1, 2, \cdots, m + 1 \\ j &= 1, 2, \cdots, m \end{aligned}\right. \end{aligned}$$

We illustrate for $m = 6$:

$$C' = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 & +1 & 0 \\ 0 & 0 & +1 & -1 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 & 0 \\ +1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & +1 & 0 & 0 \end{pmatrix} \begin{array}{c} \\ \left.\begin{array}{c} 4 \\ 2 \\ 5 \\ 3 \\ 1 \end{array}\right\} \text{Spacing between } -1 \text{ and } +1 \text{ on the middle } m-1 \text{ rows .} \\ \\ \end{array}$$

In both matrices C and C' a cyclic permutation of the column vectors does not produce a duplication before m steps. Thus each set of $m + 1$ elements are distinct and the m sets exhaust the collection of $m(m + 1)$ multinomial coefficients adjacent to A.

It should be noted that the above construction does not yield the only possible partitioning. There exist other partitionings into subsets with the desired property in both the even and odd cases. For example, for $m = 3$, the above construction yields

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

but

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

also have the desired property.

For $m = 4$, the following is a partitioning different from that yielded by the above process:

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