SOME MORE FIBONACCI DIOPHANTINE EQUATIONS

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It is well known that the Quadratic Diophantine equation $y^2 - 5x^2 = \pm 4$ has solutions in integers <u>if</u> and <u>only if</u> $y = L_n$ and $x = F_n$, n an integer. For a proof by infinite descent see [2]. The underlying identity is

$$L_n^2 - 5F_n^2 = 4(-1)^n.$$

There are other quadratic Diophantine equations which are Fibonacci-related. In "Fibonacci to the Rescue" [1], there occurs

(1) $$x^2 + x(y - 1) - y^2 = 0.$$

The proof that solutions in positive integers are possible if and only if $x = F_{2p+1}^2$ and $y = F_{2p+1}F_{2p+2}$ appears novel.

Solve quadratic equation (1) for x. In order for x to be an integer, the quadratic discriminant

$$(y - 1)^2 + 4y^2 = k^2.$$

Set $y - 1 = m^2 - n^2$, $2y = 2mn$, and $k = m^2 + n^2$ so that

$$m^2 - mn - n^2 = -1,$$

which, when solved for m yields

$$m = \frac{n \pm \sqrt{5n^2 - 4}}{2} .$$

Thus m is an integer if and only if $5n^2 - 4 = s^2$. It follows that $n = F_{2p+1}$ and $s = L_{2p+1}$ for some integer p.

It follows that $m = F_{2p+2}$ or $-F_{2p}$ since $L_{2p+1} = F_{2p+1} + 2F_{2p}$.

Thus $y = mn = F_{2p+2}F_{2p+1}$ or $-F_{2p+1}F_{2p}$. Since $k = m^2 + n^2$, it follows that, for

$$m = F_{2p+2} \quad \text{and} \quad n = F_{2p+1},$$

$$x = -F_{2p+2}^2 \quad \text{or} \quad F_{2p+1}^2 \quad \text{and} \quad y = F_{2p+2}F_{2p+1},$$

while for $m = -F_{2p}$, $n = F_{2p+1}$,

$$x = -F_{2p}^2 \quad \text{or} \quad F_{2p+1}^2 \quad \text{and} \quad y = -F_{2p+1}F_{2p} .$$

These are the only integral solutions to $x^2 + x(y - 1) - y^2 = 0$.

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