GENERAL IDENTITIES FOR RECURRENT SEQUENCES OF ORDER TWO

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1. INTRODUCTION

Let $W_0$, $W_1$, $a \neq 0$, and $b \neq 0$ be arbitrary real numbers, and define

(1.1) $W_{n+2} = aW_{n+1} - bW_n$, $a^2 - 4b \neq 0$, $(n = 0, 1, \cdots)$,

(1.2) $U_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ $(n = 0, 1, \cdots)$,

(1.3) $V_n = \alpha^n + \beta^n$ $(n = 0, 1, \cdots)$,

(1.4) $W_n = (W_0V_n - W_n)/b^n$ $(n = 0, 1, \cdots)$,

where $\alpha \neq \beta$ are roots of $x^2 - ax + b = 0$. If $W_0 = 0$ and $W_1 = 1$, then $W_n = U_n$, $n = 0, 1, \cdots$; and if $W_0 = 2$ and $W_1 = a$, then $W_n = V_n$, $n = 0, 1, \cdots$. Our first result is

**Theorem 1.** Let $W_n$ and $W^*_n$ be solutions of (1.1). Let $r$, $m$, and $n$ be integers ($+, -$, or $0$). Then, for $k = 0, 1, \cdots$,

(1.5) $\sum_{i=0}^{k} (-1)^i \binom{k}{i} W^{k-i}_{r+m} W^i_r W^*_n W^{r+k+i}_n$

\[= b^{r+k} U^k_{m} \sum_{j=0}^{k} (-1)^j \binom{k}{j} W^{k-j}_{r} W^j_0 W^*_n W^{r+k+i-j}_n \]

Special cases of (1.5) are given by

**Corollary 1.**

(1.6) $\sum_{i=0}^{k} (-1)^i \binom{k}{i} U^{k-i}_{r+m} U^i_r U^{r+k+i}_n = b^{r+k} U^k_{m} U^*_n$,
(1.7) \[ \sum_{i=0}^{k} (-1)^i \binom{k}{i} u^{k-i}_{r+m} v^i_r w_{n+rk+im} = b^{rk} u^k_m w_n, \]

(1.8) \[ \sum_{i=0}^{k} (-1)^i \binom{k}{i} u^{k-i}_{r+m} v^i_r v_{n+rk+im} = b^{rk} v^k_m v_n, \]

(1.9) \[ \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} v^{2k-i}_{r+m} v^i_r w_{n+2kr+im} = (a^2 - 4b)^k b^{2kr} u^{2k}_m w_n, \]

(1.10) \[ \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} v^{2k-i}_{r+m} v^i_r w_{n+2kr+im} = (a^2 - 4b)^k b^{2kr} u^{2k}_m w_n, \]

(1.11) \[ \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} v^{2k-i}_{r+m} v^i_r w_{n+2kr+im} = (a^2 - 4b)^k b^{2kr} u^{2k}_m v_n, \]

(1.12) \[ \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} v^{2k+1-i}_{r+m} v^i_r v_{n+(2k+1)r+im} = -(a^2 - 4b)^{k+1} b^{(2k+1)r} u^{2k+1}_m u_n, \]

Our next result related to Theorem 1 is

**Theorem 2.** Let \( W_n \) be a solution of (1.1). Let \( r, m, \) and \( n \) be integers \((+, -, 0)\). Then, for \( k = 0, 1, \ldots \), we have
$V_{kr+n}^k W_{kr+n}^k$

$$= \sum_{j=0}^{[k/2]} \binom{k}{2j} (a^2 - 4b)^j b^{2jm} V_{r+m}^{k-2j} U_r^{2j} (W_1 U_{n-2jm} - b W_0 U_{n-2jm} - 1)

+ \sum_{j=0}^{[k-1]/2} \binom{k}{2j+1} (a^2 - 4b)^j b^{2jm} V_{r+m}^{k-2j-1} U_r^{2j+1} (W_1 V_{n-2jm} - W_0 V_{n-2jm} - 1)$$

(1.14)

$$V_{kr+n}^k U_{kr+n}^k = \sum_{j=0}^{[k/2]} \binom{k}{2j} (a^2 - 4b)^j b^{2jm} V_{r+m}^{k-2j} U_r^{2j} V_{n-2jm}

+ \sum_{j=0}^{[k-1]/2} \binom{k}{2j+1} (a^2 - 4b)^j b^{2jm} V_{r+m}^{k-2j-1} U_r^{2j+1} V_{n-2jm}$$

(1.15)

$$V_{kr+n}^k V_{kr+n}^k = \sum_{j=0}^{[k/2]} \binom{k}{2j} (a^2 - 4b)^j b^{2jm} V_{r+m}^{k-2j} U_r^{2j} V_{n-2jm}

+ \sum_{j=0}^{[k-1]/2} \binom{k}{2j+1} (a^2 - 4b)^j b^{2jm} V_{r+m}^{k-2j-1} U_r^{2j+1} U_{n-2jm}$$

(1.16)

2. PROOF OF THEOREM 1

Let $W_n^* = S_1 \alpha^n + S_2 \beta^n$ and $W_n = C_1 \alpha^n + C_2 \beta^n$, $n = 0, 1, \ldots$, where $S_1$ and $C_1$, $i = 1, 2$, are arbitrary constants. Since $W_0 = C_1 + C_2$ and $W_1 = C_1 \alpha + C_2 \beta$, we readily find that

$$= \sum_{j=0}^{[k/2]} \binom{k}{2j} (a^2 - 4b)^j b^{2jm} V_{r+m}^{k-2j} U_r^{2j} V_{n-2jm}$$

(2.1) $(\alpha - \beta)^k C_1^k = (W_1 - \beta W_0)^k = \sum_{j=0}^{[k/2]} \binom{k}{2j} (a^2 - 4b)^j b^{2jm} V_{r+m}^{k-2j} U_r^{2j} V_{n-2jm}$

(2.2) $(\alpha - \beta)^k C_2^k = (\alpha W_0 - W_1)^k = (-1)^k \sum_{j=0}^{[k/2]} \binom{k}{2j} (a^2 - 4b)^j b^{2jm} V_{r+m}^{k-2j} U_r^{2j} V_{n-2jm}$
Let $L$ denote the left-hand side of (1.5). Then, using the binomial theorem and representation of $W_n^*$, we have

$$(2.3) \quad L = S_2\alpha^{n+r}W_{r+m}^* - \alpha^m W_r^* + S_2\beta^{n+r}W_{r+m}^* - \beta^m W_r^*.$$ 

Since

$$W_{r+m}^* - \alpha^m W_r^* = (\beta^m - \alpha^m)\beta^r C_2$$

and

$$W_{r+m}^* - \beta^m W_r^* = (\alpha^m - \beta^m)\alpha^r C_1,$$

we obtain, using $\alpha\beta = b$, (2.1), (2.2), and (1.2),

$$(2.4) \quad L = S_2 b^r \alpha^n (\beta^m - \alpha^m) C_2^k + S_2 b^r \beta^n (\alpha^m - \beta^m) C_1^k$$

$$= b^r \sum_{j=0}^{k} (-1)^{k-j} j^k W_1^{k-j} W_0^j (S_1\alpha^n)^j + S_2 \beta^n (S_1\alpha^n)^j + S_2 \beta^n (S_1\beta^n)^j = R,$$

where $R$ denotes the right-hand side of (1.5).

If $W_r^* = U_r$ and $W_n^* = U_n^*$, then $W_0 = 0$ and (1.5) gives the special case (1.6), noting that all terms in the right-hand sum of (1.5) vanish except for $j = 0$.

Since

$$(2.5) \quad W_n = W_0 U_n + (W_1 - a W_0) U_n,$$

we obtain (1.7) from (1.6); and (1.8) from (1.7) when $W_n^* = V_n^*$.

If $W_n = V_n^*$ (i.e., $C_1 = C_2 = 1$), then (2.4), with $k = p$, gives
Noting that \((\alpha - \beta)^2 = a^2 - 4b\), then (2.6), for \(W_n^* \equiv U_n\) (i.e., \(S_1 = -S_2 = \frac{1}{(\alpha - \beta)}\)), gives (1.9) for \(p = 2k\) and (1.13) for \(p = 2k + 1\). Using (2.5), we get (1.10) from (1.9); and (1.11) from (1.10) when \(W_n \equiv V_n\).

If \(W_n^* \equiv V_n\) (i.e., \(S_1 = S_2 = 1\)), then (2.6) gives (1.12) for \(p = 2k + 1\).

If \(a = -b = 1\), then \(U_n \equiv F_n\), and (1.6) gives the identity of Halton [1, p. 34] as a special case.

### 3. PROOF OF THEOREM 2

Our method is a generalization of a proof used in the unpublished Master's thesis of Vinson [2, pp. 14-16]. If we treat \(\alpha^r\) and \(\beta^r\) as the unknowns in the system \((\alpha - \beta)U_r = \alpha^r - \beta^r\) and \(V_{r+m} = \alpha^m \alpha^r + \beta^m \beta^r\), we obtain

\[
V_m \alpha^r = V_{r+m} + (\alpha - \beta) \beta^m U_r \quad \text{and} \quad V_m \beta^r = V_{r+m} - (\alpha - \beta) \alpha^m U_r.
\]

Since \(W_{kr+n} = C_1 \alpha^n (\alpha^r)^k + C_2 \beta^n (\beta^r)^k\), we obtain

\[
V_m W_{kr+n} = C_1 \alpha^n (V_{r+m} + (\alpha - \beta) \beta^m U_r)^k + C_2 \beta^n (V_{r+m} - (\alpha - \beta) \alpha^m U_r)^k
\]

(3.1)

\[
= \sum_{i=0}^{k} \binom{k}{i} (\alpha - \beta)^i \frac{V_{r+m} + (\alpha - \beta) \beta^m U_r}{r+m} (\alpha \beta)^{mi} \left( C_1 \alpha^{n-mi} + (-1)^i C_2 \beta^{n-mi} \right).
\]

Now

\[
C_1 \alpha^{n-mi} + (-1)^i C_2 \beta^{n-mi}
\]

\[
= \left[ (W_1 - \beta W_0) \alpha^{n-mi} + (-1)^i (\alpha W_0 - W_1) \beta^{n-mi} \right] / (\alpha - \beta)
\]

\[
= W_1 \frac{\alpha^{n-mi} - (-1)^i \beta^{n-mi}}{\alpha - \beta} - b W_0 \left( \alpha^{n-mi-1} - (-1)^i \beta^{n-mi-1} \right).
\]
Since \((a - \beta)^2 = a^2 - 4b\), we obtain (1.14) from (3.1) for \(i = 2j\) and \(i = 2j + 1\).

If \(W_n \equiv U_n\), then \(W_0 = 0, W_1 = 1\), and thus (1.14) gives (1.15). If \(W_n \equiv V_n\), then \(W_0 = 2, W_1 = a\), and thus (1.14) gives (1.16), noting that \(V_n = aU_n - 2bU_{n-1}\) and that

\[aV_n - 2bV_{n-1} = 2V_{n+1} - aV_n = (a^2 - 4b)U_n.\]

4. EXTENDED RESULTS

Our next class of results are of a higher level order than Theorem 1, since we now essentially replace \(W_n^*\) in (1.5) by its cross-product with itself.

Theorem 3. Let \(W_n\) and \(W_n^*\) be solutions of (1.1). Let \(r, m, p,\) and \(n\) be integers \((+, -, or 0)\). Then, for \(k = 0, 1, \ldots\),

\[
\left(\sum_{i=0}^{k} (-1)^i \binom{k}{i} W_r^i \left(\frac{W_n^*}{r+2m} W_{r+m}^i W_{r+n}^* W_{r+n+m}^*\right) (a^2 - 4b)\right)
\]

\[
(4.1) = br^k u_{2m} \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^j W_0^{k-j} (W_1^2 V_{p+n-rk+j} - 2bW_0^* W_{p+n-rk+j} + b^2 W_0^* W_{p+n-rk+j+2})
\]

\[-(W_1^2 - aW_0^* W_1^* + bW_0^2) b^n V_{p+n} \left(W_1 V_{r+m} - bW_0 V_{r+m-1}\right)^k.\]

Corollary 3. In special cases of (4.1), we have

\[
\sum_{i=0}^{k} (-1)^i \binom{k}{i} W_r^i V_{r+2m}^i V_{r+n+m}^i V_{r+n+m}^*
\]

\[
(4.2) = b^r u_{2m} \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^j W_0^{k-j} V_{p+n-rk+j}
\]

\[+ b^n V_{p-n} u_m^k (W_1 V_{r+m} - bW_0 V_{r+m-1})^k,\]
\[
\sum_{i=0}^{k} (-1)^i \binom{k}{i} u^{k-i}_{r+2m} v^i_{r+p+im} v^{n+im}
\]
(4.3)

\[
= b^r k u^k_{2m} v^{p+n-rk} + b^n v^{k}_{p-n} u^k_{m} v^{r+m} ,
\]

\[
\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} v^{2k-i}_{r+2m} v^i_{r+p+im} v^{n+im}
\]
(4.4)

\[
= (a^2 - 4b) k b^r k u^k_{2m} v^{p+n-2rk} + (a^2 - 4b) b^n v^{k}_{p-n} u^k_{m} v^{r+m} ,
\]

\[
\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} v^{2k+1-i}_{r+2m} v^i_{r+p+im} v^{n+im}
\]
(4.5)

\[
= -(a^2 - 4b) b^{k+1}_r (2k+1) u^{2k+1}_{2m} v^{p+n-r(2k+1)} + (a^2 - 4b) b^n v^{k+1}_{p-n} u^{2k+1}_{m} v^{r+m} ,
\]

\[
(a^2 - 4b) \sum_{i=0}^{k} (-1)^i \binom{k}{i} w^{k-i}_{r+2m} w^i_{r+p+im} w^{n+im}
\]
(4.6)

\[
= b^r k u^k_{2m} \sum_{j=0}^{k} (-1)^j \binom{k}{j} w^{k-j}_{r} w^{j}_{0} v^{p+n-rk+j} - b^n v^{k}_{p-n} u^k_{m} (w^{k}_{r} v^{r+m} - b w^{k}_{0} v^{r+m-1})^k ,
\]

\[
(a^2 - 4b) \sum_{i=0}^{k} (-1)^i \binom{k}{i} u^{k-i}_{r+2m} u^i_{r} u^{n+im} u^{n+im}
\]
(4.7)

\[
= b^r k u^k_{2m} v^{p+n-rk} - b^n v^{k}_{p-n} u^k_{m} v^{r+m} ,
\]
$$\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} r^{2k-1} r^i V^i U^r p + i m U^{n + i m}$$

(4.8)

$$= (a^2 - 4b)^{k-1} b^{2k} r^{2k} U^{2k} V^{2k} p + n - 2r k - (a^2 - 4b)^{2k-1} b^n V^{p - n} (U_m U_{p + m}^2)^{2k}$$

Closely associated with Theorem 3 is

**Theorem 4.** Let $W_n$ be a solution of (1.1). Let $r$, $m$, $p$, and $n$ be integers ($+$, $-$, or $0$). Then, for $k = 0, 1, \ldots$,

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} W^{k-i} V^i W^{r+2i} V^r U^{p+i m} V^{n+i m}$$

(4.10)

$$= b^{r k} U^{2m} \sum_{j=0}^{k} (-1)^j \binom{k}{j} W^{k-j} W^i U^{p-n-rk+j}$$

$$+ b^n U^{p-n} U^{k} (W_i V^{r+m} - b W_0 V^{r+m-1})^k$$

**Corollary 4.** As special cases of (4.10), we have

$$\sum_{i=0}^{k} (-1)^i \binom{k}{i} U^{k-i} V^i W^{r+2i} V^r U^{p+i m} V^{n+i m}$$

(4.11)

$$= b^{r k} U^{2m} U^{p+n-rk} + b^n U^{p-n} U^{k} V^{k}$$
\[ \sum_{i=0}^{2k} (-1)^i \binom{2k-i}{i} V_r^{2k-i} U_{r+2m}^i V_{n+im} \]

\[ = (a^2 - 4b)^k b^{2r} U_{2m}^{2k} U_{p+n-2r} + (a^2 - 4b)^k b^n U_{p-n} U_{m r+m}^{2k} U_{r+m}^{3k} , \]

\[ \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1-i}{i} V_r^{2k+1-i} U_{r+2m}^i V_{n+im} \]

\[ = -(a^2 - 4b)^k b^{r(2k+1)} U_{2m}^{2k+1} V_{p+n-r(2k+1)} + (a^2 - 4b)^{2k+1} b^n U_{p-n} U_{m r+m}^{2k+1} U_{r+m} . \]

**Remarks.** Since \( U_{2n} = U_n V_n \), we note that for \( p = n \), (4.10), (4.11), (4.12) and (4.13) reduce to special cases, respectively, of (1.5), (1.6), (1.9), and (1.13).

5. **PROOF OF THEOREM 3**

We readily find that

\[ W^*_p V_{n+im} = S_1^2 \alpha^{p+n} \beta^{2mi} + S_2^2 \beta^{p+n} \beta^{2mi} + S_1 S_2 b^n V_{p-n} b^{mi} . \]

Let \((a^2 - 4b) \cdot L\) denote the left-hand side of (4.1). Then, the binomial theorem, using (5.1), gives

\[ L = S_1^2 \alpha^{p+n} (W_{r+2m}^{2m} W_r^k) + S_2^2 \beta^{p+n} (W_{r+2m}^{2m} W_r^k) + S_1 S_2 b^n V_{p-n} (W_{r+2m}^{2m} W_r^k) . \]

Since \( W_n = C_1 \alpha^n + C_2 \beta^n \), we have, using (2.1) and (2.2) for \( k = 1 \),

\[ (W_{r+2m} - b^m W_r)^k = U_{m}^k (W_V r+m - b W_0 V_{r+m-1}^k) \equiv Y . \]

Noting the relations cited after (2.3), we have
Recalling (2.1) and (2.2), we now have

$$L = b^{rk} \frac{U}{2m} \sum_{j=0}^{k} (-1)^j \binom{k}{j} W_1^{k-j} W_2^{j} (S_1^2 \beta^{n-rk+j} + S_2^2 \beta^{n-rk}) + S_1 S_2 b^n V_{p-n} Y.$$

(5.5)

Since \((\alpha - \beta)S_1 = W_1^* - \beta W_0^*\) and \((\alpha - \beta)S_2 = \alpha W_0^* - W_1^*\), additional simplification of (5.5), using \(\alpha \beta = b\), \(\alpha + \beta = a\), and \((\alpha - \beta)^2 = a^2 - 4b\), yields (4.1).

If \(W_0^* = V_n\), then \(W_0^* = 2\) and \(W_1^* = a\), and thus (4.1) gives (4.2), noting that

$$a^2 V_c - 4ab V_{c-1} + 4b^2 V_{c-2} = (a^2 - 4b) V_c.$$

We get (4.3) from (4.2) when \(W_n \equiv U_n\).

If \(W_n \equiv V_n\), then (4.2) gives (4.4) and (4.5), which are also obtained from (5.4), where \(S_1 = S_2 = C_1 = C_2 = 1\).

If \(W_n^* \equiv U_n\), then (4.1) gives (4.6), which gives (4.7) for \(W_n = U_n\).

If \(W_n \equiv V_n\), then (4.6) gives (4.8) and (4.9), which are also obtained from (5.4), where now \(C_1 = C_2 = 1\) and \(S_1 = -S_2 = (\alpha - \beta)^{-1}\).

6. PROOF OF THEOREM 4

We readily find that

$$U_{p+im} V_{n+im} = \frac{\alpha^{p+n}}{\alpha - \beta} \frac{\alpha^{2mi}}{\alpha - \beta} \frac{\beta^{p+n}}{\alpha - \beta} \frac{\beta^{2mi}}{\alpha - \beta} + b^n U_{p-n} b^m l.$$
Let L denote the left-hand side of (4.10). Then, the binomial theorem, using (6.1), gives

\[
L = \frac{\alpha^{p+n}}{\alpha - \beta} (W_r^{2m} - \alpha^2 W_r) + \frac{\beta^{p+n}}{\alpha - \beta} (W_r^{2m} - \beta^2 W_r) + b^n U_{p-n} W_r^{2m - 2m W_r} \\
(6.2)
\]

Using (2.1) and (2.2) in (6.2) gives the desired result (4.10).

We obtain (4.11) from (4.10), where \( W_n \equiv U_n \). If \( W_n \equiv V_n \), then (4.10) gives (4.12) and (4.13), which are also obtained from (6.3), where \( C_1 = C_2 = 1 \).

7. ADDITIONAL SUMS

Closely related in proof to the above theorems are the following results:

Theorem 5. Let \( W_n \) and \( W^*_n \) be solutions of (1.1). Let \( m, p, \) and \( n \) be integers \((+, -, \text{ or } 0)\). Then

\[
\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} W^{p+im} W^{*n-im} = b^p U_m^{2k} (a^2 - 4b)^{k-1} Z_4(m, k),
\]

where

\[
W_0 W_1 V_{n-2jm-p+1} - b W_0 W^*_1 V_{n-2jm-p} + b W^*_0 W^*_1 V_{n-2jm-p} = Z_4(m, i),
\]

\[
\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} W^{p+im} W^{*n-im} = b^p U_m^{2k+1} (a^2 - 4b)^k \cdot Z_2(m, k),
\]

where
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\[ W_0 W_i^* U_{n-(2j+1)m-p+1} - (bW_0 W_i^* + W_1 W_i^*) U_{n-(2j+1)m-p} + b W_0^* W_1 U_{n-(2j+1)m-p-1} = Z_2(m,j). \]

**Corollary 5.** As special cases of (7.1) and (7.3), we have

\[ \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} \left( \sum_{p+im} U_{n-im} = \sum_{p+im} V_{n-im} = \sum_{p+im} W_{n-im} = \sum_{p+im} X_{n-im} = \sum_{p+im} Y_{n-im} = \sum_{p+im} Z_{n-im} = \sum_{p+im} \right) \]

\[ \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (a^2 - 4b) \]

\[ \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (a^2 - 4b)^k \]

\[ \sum_{i=0}^{2k} (-1)^i \binom{2k+1}{i} (a^2 - 4b)^k \]

\[ \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (a^2 - 4b)^{k+1} \]

\[ \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (a^2 - 4b)^{k+1} \]

\[ \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (a^2 - 4b)^{k+1} \]
Theorem 6. Let $W_n$ and $W_n^*$ be solutions of (1.1). Let $m, p$, and $n$ be integers (+, -, or 0). Then, for $k = 0, 1, \ldots$,

$$
(7.12) \quad \left( \sum_{i=0}^{k} \binom{k}{i} W_{p+i} W_{n-im} \right) (a^2 - 4b) = 2^k Z_3 + b^P V^k_m Z_4 ,
$$

where

$$
(7.13) \quad Z_3 = W_1 W_1^* V_{p+n} - b(W_0 W_1 + W_0 W_1^*) V_{p+n-1} + b^2 W_0 W_0^* V_{p+n-2} \equiv Z_3(p,n)
$$

$$
(7.14) \quad Z_4 = W_0 W_1^* V_{n-km-p+1} - (bW_0 W_0^* + W_1 W_1^*) V_{n-km-p} + bW_0 W_1 V_{n-km-p-1} .
$$

Corollary 6. As special cases of (7.12), we have

$$
(7.15) \quad \left( \sum_{i=0}^{k} \binom{k}{i} U_{p+i} U_{n-im} \right) (a^2 - 4b) = 2^k V_{p+n} - b^P V^k_m V_{n-km-p} ,
$$

$$
(7.16) \quad \sum_{i=0}^{k} \binom{k}{i} V_{p+i} V_{n-im} = 2^k V_{p+n} + b^P V^k_m V_{n-mk-p} ,
$$

$$
(7.17) \quad \sum_{i=0}^{k} \binom{k}{i} U_{p+i} U_{n-im} = 2^k U_{p+n} - b^P V^k_m U_{n-mk-p} .
$$

Remarks. Special cases of (7.5), (7.6), (7.9), and (7.10) for $U_n = F_n$ and $V_n = L_n$ were given, using matrix methods, in the paper by Hoggatt and Bicknell [3].
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8. PROOF OF THEOREMS 5 AND 6

We readily find that

\[ W_{p+im} W^*_{n-im} = S_1 C_2 \alpha^{n-im} \beta^{p+im} + C_1 S_2 \alpha^{p+im} \beta^{n-im} \]
\[ + C_3 S_1 \alpha^{p+n} + C_2 S_2 \beta^{p+n} . \]

For \( r > 0 \), we obtain, using the binomial theorem,

\[ \sum_{i=0}^{r} (-1)^i \binom{r}{i} W_{p+im} W^*_{n-im} \]
\[ = S_1 C_2 \alpha^n \beta^p (1 - \alpha^{-m} \beta^m)^r + C_1 S_2 \alpha^p \beta^n (1 - \alpha^m \beta^{-m})^r \]
\[ = b^p \sum_{m=0}^{r} (\alpha - \beta)^r (S_1 C_2 \alpha^{n-mr-p} + (-1)^r C_2 S_2 \beta^{n-mr-p}) . \]

Using (2.1) and (2.2), (8.2) gives (7.1) for \( r = 2k \) and (7.3) for \( r = 2k + 1 \).

Special cases (7.5), \ldots, and (7.11) are readily obtained from (7.1) and (7.3) for the choices indicated.

Using (8.1), we readily find that

\[ \sum_{i=0}^{k} \binom{k}{i} W_{p+im} W^*_{n-im} = 2^k (C_1 S_1 \alpha^{p+n} + C_2 S_2 \beta^{p+n}) \]
\[ + b^p \sum_{m=0}^{k} (S_1 C_2 \alpha^{n-mk-p} + C_1 S_2 \beta^{n-mk-p}) . \]

Using (2.1) and (2.2), (8.3) reduces to (7.12). Special cases (7.15), \ldots, (7.17), are readily obtained from (7.12).

9. MORE SUMS

Introduction of new integer parameters requires that we redefine certain identities by notationally including parameters previously suppressed for simplicity. Thus, we define \( Z_1(m, j, p, n) \) by (7.2); \( Z_2(m, j, p, n) \) by
Using (8.1), we can obtain the following results, whose lengthy details are omitted.

Theorem 7. Let \( W_n, W^*_n \) and \( W^{**}_n \) be solutions of (1.1). Let \( m, p, n \), and \( r \) be integers (+, -, or 0). Then, we have

\[
\sum_{i=0}^{2k} (-1)^{i} \binom{2k}{i} b^{-mi} W^{**}_{r+2im} W^*_{p+im} W^*_{n-im} \quad (k \geq 0)
\]

\[
= (a^2 - 4b)^{k-1} b^{-2km} W^{**}_{m} W^{**}_{r+2km} Z_3(p,n) + b^{p+r} (a^2 - 4b)^{k-1} U^{2k}_{2m} W^{**}_{4mk} Z_2(4mk, 0, p + r - 1, n) - W^{**}_1 Z_2(4mk, 0, p + r, n),
\]

\[
\sum_{i=0}^{2k+1} (-1)^{i} \binom{2k+1}{i} b^{-mi} W^{**}_{r+2im} W^*_{p+im} W^*_{n-im} \quad (k \geq 0)
\]

\[
= (a^2 - 4b)^{k-1} b^{-m(2k+1)} U^{k+1}_{m} W^{**}_{r+m(2k+1)-1} Z_3(p,n) + b^{p+r} (a^2 - 4b)^{k-1} U^{2k+1}_{2m} W^{**}_{Z_1(m, 2k + 1, p + r - 1, n)} - W^{**}_1 Z_1(m, 2k + 1, p + r, n),
\]

\[
(a^2 - 4b) \sum_{i=0}^{k} \binom{k}{i} b^{-mi} W^{**}_{r+2im} W^*_{p+im} W^*_{n-im} \quad (k \geq 0)
\]

\[
= 2^k b^{p+r} [ W^{**}_1 Z_2(0, k, p - r, n) - b W^{**}_1 Z_2(0, k, p - r + 1, n)] + b^{p+r} U^{2m}_{2m} W^{**}_{Z_2(2mk, 0, p + r - 1, n)} - W^{**}_1 Z_2(2mk, 0, p + r, n) + b^{-mk} U^{2m}_{m} W^{**}_{Z_3(p,n)}.
\]

Remarks. As a typical special case, we get from (9.1),
\[
\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} b^{-mi} U_{r+2im} U_{p+im} U_{n-im} \quad (k > 0)
\]

\[\text{(9.4)}\]

\[= (a^2 - 4b)^k b^{-2km} U_{m} U_{r+2km} V_{p+n}
+ b^{p+r} (a^2 - 4b)^{k-1} U_{2m} U_{n-4mk-p-r} \cdot
\]

Theorem 8. Let \( W_n, W^*, \) and \( W^{**} \) be solutions of (1.1). Let \( m, p, n, q, \) and \( r \) be integers \((+, -, \text{ or } 0)\). Then

\[\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} W_{p+im} W_{q+im} W^*_{n-im} W^{*}_{r-im} \quad (k > 0)
\]

\[\text{(9.5)}\]

\[= b^p U_{2k} (a^2 - 4b)^{k-2} Z_1(m, k, p, n) Z_3(q, r)
+ b^q U_{2k} (a^2 - 4b)^{k-2} Z_1(m, k, q, r) Z_3(p, n)
+ b^{p+q} (a^2 - 4b)^{k-2} U_{2m} A(W^{n+r-p-q-4mk}) .\]

where

\[A(W^{**}) = (W_0 W_1^*)^2 W^{**} - 2W_0 W_1^* (b W_0 W_1^* + W_1 W_1^*) W_{1+1}^*
+ [b^2 (W_0 W_1^*)^2 + 4b W_0 W_1^* W_1^* + (W_1 W_1^*)^2] W_{1+1}^*
- 2b W_0 W_1^* (b W_0 W_1^* + W_1 W_1^*) W^{**} + b^2 (W_0 W_1^*)^2 W_{1-2}^{**} .\]

(9.6)

\[\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} W_{p+im} W_{q+im} W^*_{n-im} W^{*}_{r-im} \quad (k \geq 0)
\]

\[\text{(9.7)}\]

\[= b^p U_{2k+1} (a^2 - 4b)^{k-1} Z_2(m, k, p, n) Z_3(q, r)
+ b^q U_{2k+1} (a^2 - 4b)^{k-1} Z_2(m, k, q, r) Z_3(p, n)
+ b^{p+q} (a^2 - 4b)^{k-1} U_{2m} A(U^{n+r-p-q-2m(2k+1)}) .\]
Remarks. As a special case of (9.5), we have

\[
\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} U_{p+i m} U_{q+i m} U_{r+i m} U_{t+i m} \quad (k > 0)
\]

(9.8)

\[
= -b^2 U_{m} (a^2 - 4b)^{k-2} V_{n-2km-p} V_{q+r} \\
- b^2 U_{m} (a^2 - 4b)^{k-2} V_{r-2km-q} V_{p+n} \\
+ b^2 U_{m} (a^2 - 4b)^{k-2} U_{2m} \left[ V_{n+r-p-q-4mk} \right].
\]

For Fibonacci, \( F_n \), and Lucas, \( L_n \), sequences, we obtain from (9.8), with \( a = -b = 1 \),

\[
\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} F_{p+i m} F_{q+i m} F_{n+i m} F_{r+i m} \quad (k > 0)
\]

(9.9)

\[
= (-1)^{r+1} 5^{k-2} F_{m} L_{n+2km+p} L_{q-r} \\
+ (-1)^{n+1} 5^{k-2} F_{m} L_{r+2km+q} L_{p-n} + 5^{k-2} F_{m} L_{n+r+p+q+4mk}.
\]

Theorem 9. Let \( W_n, W^*_n \), and \( W^{**}_n \) be solutions of (1.1). Let \( m, p, n, q, r, \) and \( t \) be integers (+, −, or 0). Then

\[
\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} b^{-2mi} W^{**}_n U_{m} \left[ W^{**}_0 A(U_{n+r+t-p-q-2mk}) - b W^{**}_n A(U_{n+r+t-p-q-2mk-1}) \right] \\
+ b^{p+q+t} (a^2 - 4b) U_{m} \left[ W^{**}_0 A(U_{n+r+t-p-q-2mk+1}) - W^{**}_n A(U_{n+r+t-p-q-6mk}) \right] \\
+ b^{2mk} (a^2 - 4b) U_{m} \left[ W^{**}_0 A(U_{n+r+t-p-q-2mk}) - b W^{**}_n A(U_{n+r+t-p-q-2mk-1}) \right] \\
+ b^{2mk+n+q} (a^2 - 4b) U_{m} \left[ W^{**}_0 A(U_{n+r+t-p-q-2mk}) - b W^{**}_n A(U_{n+r+t-p-q-2mk-1}) \right],
\]

where
THEOREM 10. Let $W_n$ and $W^*_n$ be solutions of (1.1). Let $m, p, n, q, r, t,$ and $s$ be integers (+, −, or 0). Then, for $k > 0$,

$$
\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} b^{-m} U^{2k+1}_{2m} W_{p+i} W_{q+i} W^*_i W^*_{r-i} \quad (k \geq 0)
$$

$$
= b^{p+q} U^{2k+1}_{2m} A(V_{n+r+p-q-m(2k+1)+1}) - b^{p+q} A(V_{n+r+p-q-m(2k+1)+1})
$$

$$
+ b^{p+q} U^{2k+1}_{2m} A(V_{n+r+p-q-t-3m(2k+1)+1}) - b^{p+q} A(V_{n+r+p-q-t-3m(2k+1)+1})
$$

$$
+ b^{p+q} U^{2k+1}_{2m} A(V_{n+r+p-q-t-3m(2k+1)+1}) - b^{p+q} A(V_{n+r+p-q-t-3m(2k+1)+1})
$$

$$
+ b^{p+q} U^{2k+1}_{2m} A(V_{n+r+p-q-t-3m(2k+1)+1}) - b^{p+q} A(V_{n+r+p-q-t-3m(2k+1)+1})
$$

$$
+ b^{p+q} U^{2k+1}_{2m} A(V_{n+r+p-q-t-3m(2k+1)+1}) - b^{p+q} A(V_{n+r+p-q-t-3m(2k+1)+1})
$$

(Continued, next page.)
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\[ \sum_{i=0}^{2k} (-1)^i \binom{2k+1}{i} U_{p+i} U_{q+i} U_{s+i} U_{r+i} U_{t+i} \] (9.14)

Remarks. As a special case of (9.13), we have

\[ \sum_{i=0}^{2k} (-1)^i \binom{2k+1}{i} U_{p+i} U_{q+i} U_{s+i} U_{r+i} U_{t+i} \] (9.15)

(Continued, next page.)
10. BINOMIAL SUMS WITH TRIPLE CROSS-PRODUCTS

The following results are an extension of Theorem 3.

**Theorem 11.** Let \( W_1 \) be a solution of (1.1). Let \( m, p_1, p_2, \) and \( p_3 \) be integers \((+, -, 0)\). Let \( p_1 + p_2 + p_3 = p \). Then

\[
\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} \left( b^m V_{2m} \right)^{2k-1} V_{1m}^i \left( \sum_{j=1}^{3} W_j^{i+im} \right) \quad (k > 0)
\]

(10.1) \[
= U_{2m}^{2k} (a^2 - 4b)^{k-1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i U_{p+2mk-i}^{p}
\]

\[
= \frac{2k+1}{2m} (a^2 - 4b)^{k-1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_0)^i V_{p+m(2k+1)-i}
\]

(10.2)
where

\[
Y_I = \sum_{j=1}^{3} b_j (W_I V_{p-2p_j-m(2k+1)} - b W_0 V_{p-2p_j-m(2k+1)-1})
\]

Special cases of Theorem 11, with \( p_1 = p_2 = p_3 = n \), are given by

\[
\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{2m})^{2k-i} v^i_m U^3_{n+i} \quad (k > 0)
\]

\[
= (a^2 - 4b)^{k-1} (U_{3m} V_{3n+2mk} - 3b^{n+3mk} U_{m}^{2k} V_{n-2mk})
\]

\[
\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m V_{2m})^{2k-i} v^i_m U^3_{n+i} \quad (k > 0)
\]

\[
= (a^3 - 4b)^k (U_{3m} V_{3n+2mk} + 3b^{n+3mk} U_{m}^{2k} V_{n-2mk})
\]

\[
\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{2m})^{2k+1-i} v^i_m U^3_{n+i} \quad (k \geq 0)
\]

\[
= (a^2 - 4b)^{k-1} (-U_{3m} V_{3n+m(2k+1)} + 3b^{n+2m(2k+1)} U_{m}^{2k+1} V_{n-m(2k+1)})
\]

\[
\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{2m})^{2k+1-i} v^i_m U^3_{n+i} \quad (k \geq 0)
\]

\[
= (a^3 - 4b)^{k+1} (-U_{3m} V_{3n+m(2k+1)} - 3b^{n+2m(2k+1)} U_{m}^{2k+1} V_{n-m(2k+1)})
\]

11. BINOMIAL SUMS WITH FOUR CROSS-PRODUCTS

The following results are an extension of Theorem 3.
Theorem 12. Let $W_n$ be a solution of (1.1). Let $m$, $p_1$, $p_2$, $p_3$, and $p_4$ be integers (+, -, or 0). Let $p_1 + p_2 + p_3 + p_4 = p$. Then

$$
(11.1) \quad \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} \left( b^m v_{3m} \right)^{2k-i} v_i^j \left( \prod_{j=1}^{4} W_{p_j + im} \right) (k > 0)
$$

$$
= U_{4m}^2 (a^2 - 4b)^{k-2} \sum_{i=0}^{4} (-1)^i \binom{4}{i} W_{i}^4 (bW_0)^i v_{p+2mk-i} + U_{2m}^2 D(W_0, W_1)(a^2 - 4b)^{2k-2} \sum_{i=1}^{3} b^{2mk+4p_1+i} v_{p-2p_1-2p_1+i}
$$

$$
+ U_{2m}^2 D(W_0, W_1)(a^2 - 4b)^{k-2} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_{i}^2 (bW_0)^i \sum_{j=1}^{4} b^{4mk+4p_j} v_{p-2p_j-2mk-1}
$$

$$
(11.2) \quad \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} \left( b^m v_{3m} \right)^{2k+1-i} v_i^j \left( \prod_{j=1}^{4} W_{p_j + im} \right) (k \geq 0)
$$

$$
= -U_{4m}^{2k+1} (a^2 - 4b)^{k-1} \sum_{i=0}^{4} (-1)^i \binom{4}{i} W_{i}^4 (bW_0)^i U_{p+m(2k+1)-1} + (U_{m} U_{2m})^2 D(W_0, W_1)(a^2 - 4b)^{2k-1} \sum_{i=1}^{3} b^{m(2k+1)+4p_1+i} v_{p-2p_1-2p_1+i}
$$

$$
- U_{2m}^{2k+1} D(W_0, W_1)(a^2 - 4b)^{k-1} \cdot \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_{i}^2 (bW_0)^i \sum_{j=1}^{4} b^{2m(2k+1)+4p_j} v_{p-2p_j-m(2k+1)-i}
$$

Special cases of Theorem 12, with $p_1 = p_2 = p_3 = p_4 = n$, are given by
The following results are companion results for Theorems 11 and 12.
Theorem 13. Let \( W \) be a solution of (1.1). Let \( m, p_1, p_2, \) and \( p_3 \) be integers (+, -, or 0). Let \( p_1 + p_2 + p_3 = p \). Then

\[
\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m v_{2m})^{2k-i} v_m \left( v_{p_1+im} \prod_{j=2}^{3} W_{p_j+im} \right) \quad (k > 0)
\]

\[
= U_{3m}^{2k} (a^2 - 4b)^{k-1} \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} v_m W_1^{2-i} (bW_0)^i v_{p-2p_j-2mk-1}
\]

(12.1)

\[
+ D(W_0, W_1) b^{4mk} U_{2m}^{2k} (a^2 - 4b)^{k-1} \sum_{j=2}^{3} b_j W_{p-j-2mk}
\]

\[
+ U_m^{2k} (a^2 - 4b)^{k-1} b^{4mk+p_1} \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} W_1^{2-i} (bW_0)^i v_{p-2p_j-2mk-1}
\]

\[
\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m v_{2m})^{2k+1-i} v_m \left( v_{p_1+im} \prod_{j=2}^{3} W_{p_j+im} \right) \quad (k \geq 0)
\]

\[
= -U_{3m}^{2k+1} (a^2 - 4b)^{k} \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} W_1^{2-i} (bW_0)^i U_{p+m(2k+1)-i}
\]

(12.2)

\[
- D(W_0, W_1) b^{2m(2k+1)} U_{m}^{2k+1} (a^2 - 4b)^{k} \sum_{j=2}^{3} b_j U_{p-2p_j-m(2k+1)}
\]

\[
- U_m^{2k+1} (a^2 - 4b)^{k} b^{2m(2k+1)+p_1} \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} W_1^{2-i} (bW_0)^i U_{p-2p_j-m(2k+1)-i}
\]
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\[
\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m v_{2m})^{2k-i} v^i_m \left( U_{p_i+m} \prod_{j=2}^{3} W_{p_j+m} \right) \quad (k > 0)
\]

\[
= U^{2k}_{3m} (a^2 - 4b)^{k-1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (b W_0)^i U_{p+2mk-1}
\]

(12.3)

\[
+ U^{2k}_{m} D(W_0, W_1) (a^2 - 4b)^{k-1} \sum_{j=2}^{3} b^{4mk+p_j} U_{p-2p_j-2mk}
\]

\[
- (a^2 - 4b)^{k-1} U^{2k}_{m} b^{4mk+p_1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (b W_0)^i U_{p-2p_1-2mk-1}
\]

\[
\sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m v_{2m})^{2k-i} v^i_m \left( U_{p_i+m} \prod_{j=2}^{3} V_{p_j+m} \right) \quad (k > 0)
\]

(12.4)

\[
= (a^2 - 4b)^k U^{2k}_{3m} U_{p+2mk}
\]

\[
+ U^{2k}_{m} (a^2 - 4b)^k \sum_{j=2}^{3} b^{4mk+p_j} U_{p-2p_j-2mk}
\]

\[
- (a^2 - 4b)^k U^{2k}_{m} b^{4mk+p_1} U_{p-2p_1-2mk}
\]

\[
\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m v_{2m})^{2k+1-i} v^i_m \left( U_{p_i+m} \prod_{j=2}^{3} W_{p_j+m} \right) \quad (k \geq 0)
\]

\[
= -U^{2k+1}_{3m} (a^2 - 4b)^{k-1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{i} (b W_0)^i U_{p+m(2k+1)-1}
\]

(12.5)

\[
- U^{2k+1}_{m} D(W_0, W_1) (a^2 - 4b)^{k-1} \sum_{j=2}^{3} b^{2m(2k+1)+p_j} V_{p-2p_j-m(2k+1)}
\]

\[
+ (a^2 - 4b)^{k-1} U^{2k+1}_{m} b^{2m(2k+1)+p_1} \sum_{i=0}^{2} (-1)^i \binom{2}{i} W_1^{2-i} (b W_0)^i U_{p-2p_1-m(2k+1)-1}
\]
\[ \sum_{j=0}^{2k+1} (-1)^j \binom{2k+1}{j} (b^{m_{V_{2m}}} V_{3m})^{2k+1-j} V_{m}^i \left( \frac{3}{p_1+i m} \prod_{j=2}^{4} \frac{1}{p_j+i m} \right) \quad (k \geq 0) \]

\[ = -U^{2k+1}_{3m} (a^2 - 4b)^k V_{p+i m(2k+1)} \]

(12.6)

\[ - U^{2k+1}_{m} (a^2 - 4b)^k \sum_{j=2}^{3} b^{2m(2k+1)+p_j} V_{p-2p_j-m(2k+1)} \]

\[ + (a^2 - 4b)^k U^{2k+1}_{m} b^{2m(2k+1)+p_1} V_{p-2p_1-m(2k+1)} \]

Theorem 14. Let \( W_n \) be a solution of (1.1). Let \( m, p_1, p_2, p_3, \) and \( p_4 \) be integers \((+, -\), or 0). Let \( p_1 + p_2 + p_3 + p_4 = p \). Then

\[ \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^{m_{V_{2m}}} V_{3m})^{2k-i} V_{m}^i \left( \frac{4}{p_1+i m} \prod_{j=2}^{4} \frac{1}{p_j+i m} \right) \quad (k > 0) \]

\[ = (a^2 - 4b)^k U^{2k-1}_{4m} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_{4}^{3-i}(b W_{9})^i U_{p+2mk+i} \]

(12.7)

\[ + D(W_6, W_4)(a^2 - 4b)^{2k-1}(U_{m} U_{2m})^{2k} \sum_{i=1}^{3} b^{2mk+p_1+p_1+i} W_{p-2p_1-2p_1+i} \]

\[ + D(W_6, W_4)(a^2 - 4b)^{k-1} U^{2k}_{2m} \sum_{j=2}^{4} b^{4mk+p_j} W_{p-2p_j-2mk} \]

\[ + (a^2 - 4b)^{k-1} U^{2k}_{2m} b^{4mk+p_1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_{4}^{3-i}(b W_{9})^i U_{p-2p_1-2mk-i} \]
\[
\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m v_3^m)^{2k+1-i} v_m^i \left( \prod_{j=2}^4 w_{p_j+i} \right) (k \geq 0)
\]

\[
= -(a^2 - 4b)^{k-1} u_{2k+1} \sum_{i=0}^{2k+1} (-1)^i \binom{3}{i} w_1^{3-i} (b w_0)^i v_{p+m(2k+1)-i}
\]

(12.8)

\[+ D(W_0, W_1)(a^2 - 4b)^{2k} (u_m u_{2m}) \sum_{i=1}^{2m} b^m (2k+1) + p_1 + p_1 + w_{p-2p_1 - 2p_1 + i}\]

\[- (a^2 - 4b)^{k-1} u_{2k+1} b^{2m(2k+1) + p_1} \sum_{i=0}^{2m} (-1)^i \binom{3}{i} w_1^{3-i} (b w_0)^i v_{p-2p_1 - m(2k+1) - i}\]

\[- (a^2 - 4b)^{k-1} D(W_0, W_1) u_{2k+1} b^{2m(2k+1)} y_2(2k + 1) \]

where

(12.9)

\[y_2(k) = \sum_{j=2}^{4} b^j (w_1 v_{p-2p_j - mk} - b w_0 v_{p-2p_j - mk-1}) \]

\[+ \sum_{i=0}^{2k} (-1)^i \binom{2k}{i} (b^m v_3^m)^{2k-i} v_m^i \left( \prod_{j=1}^2 v_{p_j+i} \right) \left( \prod_{j=3}^4 w_{p_j+i} \right) \quad (k > 0)
\]

(12.10)

\[= (a^2 - 4b)^{k-1} u_{2k} \sum_{i=0}^{2k} (-1)^i \binom{2}{i} w_1^{2-i} (b w_0)^i v_{p+2mk-i}
\]

\[+ (a^2 - 4b)^{2k} (u_m u_{2m}) b^{2mk+p_1+p_2} \sum_{i=0}^{2k} (-1)^i \binom{2}{i} w_1^{2-i} (b w_0)^i v_{p-2p_1 - 2p_2 - i}
\]

\[+ D(W_0, W_1)(a^2 - 4b)^{2k-1} (u_m u_{2m}) b^{2mk+p_1+p_2+j} \sum_{j=1}^{2k} v_{p-2p_1 - 2p_2 - j}
\]

(Continued, next page.)
\begin{equation}
+ (a^2 - 4b)^{k-1} U^{2k}_{2m} \sum_{j=1}^{2} b^{4m \cdot p_j} \sum_{i=0}^{2} (-1)^i \left( \begin{array}{c} 2k \\ i \end{array} \right) W_1^{2-i} (bW_0)^i V_{p-2p_j-2mk-1}
\end{equation}

+ (a^2 - 4b)^{k-1} D(W_0, W_1) U^{2k}_{2m} \sum_{j=3}^{4} b^{4m \cdot p_j} V_{p-2p_j-2mk} \tag{12.11}

\begin{equation}
\sum_{i=0}^{2k+1} (-1)^i \left( \begin{array}{c} 2k + 1 \\ i \end{array} \right) \left( b^m V_{3m} \right)^{2k+1-i} V^i_m \left( \prod_{j=1}^{2k+1-i} W_{p+j} + i \right) \left( \prod_{j=3}^{4} W_{p_j} + i \right)
\end{equation}

\begin{equation}
- (a^2 - 4b)^k U^{2k+1}_{2m} \sum_{j=1}^{2} b^{2m(2k+1) \cdot p_j} \sum_{i=0}^{2} (-1)^i \left( \begin{array}{c} 2k \\ i \end{array} \right) W_1^{2-i} (bW_0)^i V_{p-2p_j-m(2k+1)-i}
\end{equation}

\begin{equation}
- (a^2 - 4b)^k U^{2k+1}_{2m} D(W_0, W_1) \sum_{j=3}^{4} b^{2m(2k+1) \cdot p_j} V_{p-2p_j-m(2k+1)} \tag{12.12}
\end{equation}

\begin{equation}
\sum_{i=0}^{2k} (-1)^i \left( \begin{array}{c} 2k \\ i \end{array} \right) \left( b^m V_{3m} \right)^{2k-i} V^i_m \left( U_{p_1+i} \prod_{j=2}^{4} W_{p_j} + i \right) \tag{k > 0}
\end{equation}

\begin{equation}
= (a^2 - 4b)^{k-2} U^{2k}_{4m} \sum_{i=0}^{3} (-1)^i \left( \begin{array}{c} 3 \\ i \end{array} \right) W_{3-i}^3 (bW_0)^i V_{p+2mk-1}
\end{equation}

(Continued, next page.)
- \((a^2 - 4b)^{2k-2}D(W_0, W_1)(U_{m2m})^2k\sum_{j=1}^{3} b^{2mk+p_1+p_1+j}(W_jV_p-2p_1-2p_1+j)
- \text{bW}_0V_p-2p_1-2p_1+j+1)\)
\[-(a^2 - 4b)^{k-2}U_{2m}^2b^{4mk+p_1}\sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i}(bW_0)^iV_p-2p_1-2mk-1)\)
\[+(a^2 - 4b)^{k-2}U_{2m}^2b^{4mk+p_1}(W_jV_p-2p_j-2mk) - \text{bW}_0V_p-2p_j-2mk-1)\)

\[(12.13) \sum_{i=0}^{2k} \text{(-1)}^i \binom{2k}{i} b^{mV_3m}^2k-1V_m^1 \left(U_{p+i+m} \sum_{j=2}^{4} V_{p_j+i+m} \right) (k > 0)\]
\[= (a^2 - 4b)^kU_{4m}^2b^{p+2mk}\]
\[-(a^2 - 4b)^{2k-2}U_{m2m}^2b^{2mk+p_1+p_1+j}U_{p-2p_1-2p_1+j}\]
\[-(a^2 - 4b)^{k}U_{m}^2b^{4mk+p_1}U_{p-2p_1-2mk}\]
\[+(a^2 - 4b)^kU_{2m}^4b^{4mk+p_1}U_{p-2p_j-2mk} \]

\[(12.14) \sum_{i=0}^{2k+1} \text{(-1)}^i \binom{2k+1}{i} b^{mV_3m}^{2k+1-i}V_m^1 \left(U_{p+i+m} \sum_{j=2}^{4} W_{p_j+i+m} \right) (k \geq 0)\]
\[= -(a^2 - 4b)^{k-1}U_{4m}^2b^{2k+1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i}(bW_0)^iU_{p+m(2k+1)-i}\]
\[-(a^2 - 4b)^{2k-1}D(W_0, W_1)(U_{m2m})^{2k+1} \sum_{j=1}^{3} b^{m(2k+1)+p_1+p_1+j}(W_jV_p-2p_1-2p_1+j)\]
\[-bW_0V_p-2p_1-2p_1+j+1)\)

(Continued, next page.)
- \((a^2 - 4b)^{k-1}D(W\theta, W_\phi) U_{2m}^{2k+1} \sum_{j=2}^{4} b^{2m(2k+1)+pj} W_{p-2p_j-m(2k+1)}\)

+ \((a^2 - 4b)^{k-1} U_{2m}^{2k+1} b^{2m(2k+1)+p1} \sum_{i=0}^{3} (-1)^i \binom{3}{i} W_1^{3-i} (bW_\theta)^i U_{p-2p_1-m(2k+1)-1}\)

\((12.15) \sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} (b^m V_{3m})^{2k+1-i} V_2^{i-1} U_{p+i} \prod_{j=2}^{4} \frac{V_{p_j+i}}{j} \) (\(k > 0\))

\(- (a^2 - 4b) U_{4m}^{2k+1} V_{p+m(2k+1)}\)

\(- (a^2 - 4b) U_{2m}^{2k+1} U_{m} \sum_{j=1}^{2k+1} b^{m(2k+1)+p1+j} U_{p-2p_1-2p_j+j}\)

\(- (a^2 - 4b) U_{2m}^{2k+1} \sum_{j=2}^{4} b^{2m(2k+1)+pj} V_{p-2p_j-m(2k+1)}\)

+ \((a^2 - 4b) U_{2m}^{2k+1} b^{2m(2k+1)+p1} V_{p-2p_1-m(2k+1)}\)

13. REMARKS ON THE PAPER BY CARLITZ AND FERNS [4]

All the important identities of Sections 1 and 2 of the above paper are special cases of our general results. Indeed, for the proper choices of parameters, our result, (1.7), contains as special cases, identities (1.6), (1.8), (1.10), (1.11), and (1.12) of [4, pp. 62–64]. In [4, pp. 65–66], I noted misprints and omissions in (2.9) (for \(n\) odd), (2.10) (for \(n\) even and odd), and (2.11) (for \(n\) even and odd). If these errors are corrected, we can then say, for the proper choices of parameters, our (1.9) gives their (2.8) and (2.10) for even \(n\); our (1.13) gives their (2.8) and (2.10) for odd \(n\). Also, our (1.12) gives their (2.9) and (2.11) for odd \(n\). The odd and even \(n\) refers to their identity usage, not ours. Our (1.11) gives their (2.9) and (2.11) for even \(n\).

The remaining portion of [4] obtained transformation identities for the Fibonacci and Lucas sequences as an application of Legendre and Jacobi polynomials. Under proper linear substitutions, these same polynomials could
The following pair of polynomial identities,

\begin{align*}
(13.1) \quad x^{2n+1} - (x - 1)^{2n+1} &= \sum_{i=0}^{n} \frac{2n + 1}{2i + 1} \binom{n + i}{2i} (x^2 - x)^{n-i}, \\
(13.2) \quad x^{2n+2} - (x - 1)^{2n+2} &= (2x - 1) \sum_{i=0}^{n} \frac{n + 1 + i}{2i + 1} (x^2 - x)^{n-i},
\end{align*}

appeared as a proposed problem 4356, p. 479, in the American Mathematical Monthly, 56 (1949), and their solution, in the same journal, 58 (1951), appears on pp. 268-269. We now proceed to apply (13.1) and (13.2) to obtain identities for \( U_n \) and \( V_n \) of (1.1).

Recalling that \( \alpha \) and \( \beta \) are roots of \( x^2 = ax - b \) (see (1.1)), set \( x = (a/b)y \) in (13.1) to obtain

\begin{align*}
(13.3) \quad a^{2n+1}y^{2n+1} - (ay - b)^{2n+1} &= \sum_{i=0}^{n} \frac{2n + 1}{2i + 1} \binom{n + i}{2i} b^{2i+1} (ay - b)^{n-i},
\end{align*}

Thus, (13.3) for \( y = \alpha \) and \( y = \beta \) gives the identities

\begin{align*}
(13.4) \quad a^{2n+1}U_{2n+1} - V_{4n+2} &= \sum_{i=0}^{n} \frac{2n + 1}{2i + 1} \binom{n + i}{2i} b^{2i+1} a^{n-i} V_{3n-3i}, \\
(13.5) \quad a^{2n+1}U_{2n+1} - U_{4n+2} &= \sum_{i=0}^{n} \frac{2n + 1}{2i + 1} \binom{n + i}{2i} b^{2i+1} a^{n-i} U_{3n-3i}.
\end{align*}

In (13.2), set \( x = (ay)/(2b) \) to obtain for even \( n = 2k \), noting that \( a\alpha - 2b = a^2 - b = \alpha(\alpha - \beta), \quad a\beta - 2b = \beta(\beta - \alpha) \).
GENERAL IDENTITIES FOR
RECURRENT SEQUENCES OF ORDER TWO

\[
\begin{align*}
\{a^{4k+2} - (a^2 - 4b)^{2k+1} - (a^2 - 4b)^k a^{2k}(4b)(2k + 1)\} V_{4k+2} \\
= 2 \sum_{j=0}^{k-1} (a^2 - 4b)^j a^{2j} (2b)^{4(k-j)+1} \left(\frac{4k + 1 - 2j}{2j}\right) V_{4j+2} \\
+ 2 \sum_{i=1}^{k} (a^2 - 4b)^i a^{2i-1} (2b)^{4(k-i)+3} \left(\frac{4k + 2 - 2i}{2i - 1}\right) U_{4i} .
\end{align*}
\]

(13.6)

An identity similar to (13.6) is obtained for \( n = 2k + 1 \). We note that the factor \((2x - 1)\) in (13.2) is troublesome for obtaining identities in \( U_n \) and \( V_n \) for (1.1), but is not so for the Fibonacci sequences.

Additional identities for \( U_n \) and \( V_n \) are readily obtained from (13.1). For complete generality, we note that \( \alpha \) and \( \beta \) satisfy \( x^m = U_{m-1} x + bU_{m-1} \). Thus, from (13.1), we obtain, having set \( x = (U_{m-1} y)/(bU_{m-1}) \), the general identity

\[
U_{2n+1}^m W_{2n+1+p} - W_{2mn+m+p} = \sum_{i=0}^{n} \frac{2n + 1}{2i + 1} \left(\frac{n + i}{2i}\right) U_{n-i}^m \left(bU_{m-1}\right)^{2i+1} W_{(m+1)(n-i)+p} ,
\]

(13.7)

where \( U_n \) and \( W_n \) are solutions of (1.1).

It should be noted that (13.1) gives Fibonacci identities that are not special cases of (13.7). As a partial listing, we have

\[
L_{2n+1} = \sum_{i=0}^{n} \frac{2n + 1}{2i + 1} \left(\frac{n + i}{2i}\right) ,
\]

(13.8)

\[
H_{12n+6+p} - 4^{2n+1} H_{6n+3+p} = \sum_{i=0}^{n} \frac{2n + 1}{2i + 1} \left(\frac{n + i}{2i}\right) 4^{n-i} H_{9n-9i+p} ,
\]

(13.9)

[Continued on page 421.]