ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-186 Proposed by James Desmond, Florida State University, Tallahassee, Florida.

The generalized Fibonacci sequence is defined by the recurrence relation

\[ U_{n-1} + U_n = U_{n+1} \]

where \( n \) is an integer and \( U_0 \) and \( U_1 \) are arbitrary fixed integers.

For a prime \( p \) and integers \( n, r, s \) and \( t \) show that

\[ U_{np+r} \equiv U_{sp+t} \pmod{p} \]

if \( p \equiv \pm1 \pmod{5} \) and \( n + r = s + t \), and that

\[ U_{np+r} \equiv (-1)^{r+t} U_{sp+t} \pmod{p} \]

if \( p \equiv \pm2 \pmod{5} \) and \( n - r = s - t \).


Problem: Show that a positive integer \( n \) is a Fibonacci number if and only if either \( 5n^2 + 4 \) or \( 5n^2 - 4 \) is a square.
Prove that there are no even perfect Fibonacci numbers.

**SOLUTIONS**

### A NORMAL DETERMINANT

**H-168** Proposed by David A. Klarner, University of Alberta, Edmonton, Alberta, Canada.

If

\[
a_{ij} = \begin{pmatrix} 1 + j - 2 \\ i - 1 \end{pmatrix}
\]

for \( i, j = 1, 2, \ldots, n \), show that \( \det a_{ij} = 1 \).

**Solution by F. D. Parker, St. Lawrence University, Canton, New York.**

It will be convenient to denote the given matrix by \( M_n \) and its determinant by \( d(M_n) \), and then to prove the result by mathematical induction.

Since

\[
a_{ij} = \begin{pmatrix} 1 + j - 2 \\ i - 1 \end{pmatrix},
\]

we have the two identities

\[ a_{ij} - a_{i-1,j} = a_{i,j-1}, \]

and

\[ a_{ij} - a_{i,j-1} = a_{i-1,j}. \]

If we subtract from each column (except the first) of \( M_n \) the preceding column, the second identity shows that

\[
d(M_n) = d(C_{i1}, C_{i-1,2}, C_{i-1,3}, \ldots, C_{i-1,n}),
\]
where \( c_{ij} \) represents a column whose elements are given by \( a_{ij} \). We notice that the first row of this new matrix is \((1, 0, 0, \cdots)\). Now if we subtract from each row (except the first) of the new matrix the preceding row, the first identity produces the matrix

\[
\begin{pmatrix}
1 & \bar{0} \\
1 & M_{n-1}
\end{pmatrix}
\]

where \( \bar{0} \) is a row vector of zeros, \( I \) is a column vector of ones. The determinant has not been changed by these operations so that we have

\[
d(Mn) = d(Mn') = d(Mn - 1).
\]

Thus \( d(Mn) \) is a constant and, since \( d(m1) = 1 \), then \( d(Mn) = 1 \).

Also solved by C. B. A. Peck and M. Yoder.

PRIME TARGET

H-169 Proposed by Francis DeKoven, Highland Park, Illinois. (Correction).

Show \( n^2 + 1 \) is a prime if and only if \( n \not= ab + cd \) with \( ad - bc = \pm 1 \) for integers \( a, b, c, d > 0 \).

Solution by Robert Guili, San Jose State College, San Jose, California. (Partial)

Note: \( \mathbb{Z} \) denotes the set of positive integers.

Solution by contradiction: If

\[
n = ab + cd; \quad ad - bc = \pm 1 ,
\]

then

\[
n^2 = a^2b^2 + 2abcd + c^2d^2; \quad 1 = a^2d^2 - 2abcd + b^2c^2 .
\]

So
\[ n^2 + 1 = a^2b^2 + a^2d^2 = c^2d^2 + b^2c^2 \]
\[ = a^2(b^2 + d^2) + c^2(d^2 + b^2) \]
\[ = (a^2 + c^2)(b^2 + d^2) \]

which is not true.

**EDITORIAL COMMENT**

The second part of this proof intended here was not complete. The *late* proposer made the same logical oversight. However, the second proof he submitted was more complete and can appear at a later date. 

Editor V. E. H.

Also solved by the Proposer.

**NON-EXISTENT**

H-171 Proposed by Douglas Lind, Stanford University, Stanford, California.

Does there exist a continuous real-valued function \( f \) defined on a compact interval \( I \) of the real line such that

\[
\int_I f(x)^n \, dx = F_n .
\]

What if we require \( f \) only be measurable?

*Solution by the Proposer.*

We claim that such a measurable function \( f \) does not exist. By the Binet formula,
\[ F_n = (a^n - b^n) / \sqrt{5} , \]

where

\[ a = (1 + \sqrt{5})/2, \quad b = (1 - \sqrt{5})/2 . \]

For any measurable real-valued function \( g \) defined on \( I \) and any \( p \geq 1 \) we define

\[ \|g\|_{p,I} = \|g\|_p = \left( \int_I |g(x)|^p dx \right)^{1/p} \]

which is taken to be \( +\infty \) if \( |g|^p \) is not Lebesgue integrable. Also, let

\[ \|g\|_{\infty,I} = \|g\|_{\infty} = \text{ess sup} \{ |g(x)|; \ x \in I \} = \inf \{ t; \mu(g^{-1}(t,\infty)) = 0 \} , \]

where \( \mu \) denotes Lebesgue measure on the real line. It is well known that since \( \mu (I) < \infty \),

\[ \lim_{p \to \infty} \|g\|_p = \|g\|_{\infty} , \]

where \( \|g\|_{\infty} \) is possibly \( \infty \).

Now suppose that \( f \) is a real-valued function on \( I \) such that

\[ F_n = \int_I f^p(x) \, dx \]

for \( n = 1, 2, \cdots \). Then

\[ \|f\|_{\infty} = \lim_{n \to \infty} \|f\|_n = \lim_{n \to \infty} F_n^{1/n} = a . \]

Let

\[ A = \{ x \in I: f(x) = a \} , \]

\[ B = \{ x \in I: f(x) = -a \} . \]
Then for \( n = 2k \) we have

\[
\frac{a^{2k} - b^{2k}}{\sqrt{5}} = \int_1^a f^{2k}(x)dx = \{\mu(A) + \mu(B)\}a^{2k} + \int_{I-(A\cup B)} f^{2k}(x)dx ,
\]
so that

\[
(*) \quad \frac{1}{\sqrt{5}} - \mu(A) - \mu(B) = \frac{1}{\sqrt{5}} \left( \frac{b}{a} \right)^{2k} + \int_{I-(A\cup B)} \left[ \frac{f(x)}{a} \right]^{2k} \, dx .
\]

Since \( |f(x)/a| < 1 \) for almost all \( x \in I - (A\cup B) \),

\[
\{f(x)/a\}^{2k} \to 0
\]
a.e. on \( I - (A\cup B) \) as \( k \to \infty \), so by Lebesgue’s Dominated Convergence Theorem, the right-hand integral approaches 0 as \( k \to \infty \). Since

\[
|b/a| < 1, \quad (b/a)^{2k} \to 0
\]
as \( k \to \infty \), so letting \( k \to \infty \) in (*) shows

\[
\mu(A) + \mu(B) = 1/\sqrt{5} .
\]

Now if we put \( n = 2k + 1 \), we have

\[
\frac{a^{2k+1} - b^{2k+1}}{\sqrt{5}} = \{\mu(A) - \mu(B)\}a^{2k+1} + \int_{I-(A\cup B)} f^{2k+1}(x)dx ,
\]
and the same reasoning as before shows

\[
\mu(A) - \mu(B) = 1/\sqrt{5} ,
\]
Hence \( \mu(B) = 0 \) and \( \mu(A) = 1/\sqrt{5} \). Letting \( K = I - A \), we thus have
\[
-\frac{b^n}{\sqrt{5}} = \int_{K} f^n(x)dx .
\]

Now

\[
|b| = \lim_{n \to \infty} \|f\|_{n,K} = \|f\|_{\infty,K} ,
\]

so

\[
\|f(x)\| \leq |b|
\]

for almost all \( x \in K \). Let

\[
C = \{ x \in K : f(x) = b \} ,
\]

\[
D = \{ x \in K : f(x) = -b \} .
\]

Then

\[
-\frac{b^{2k}}{\sqrt{5}} = \{ \mu(C) + \mu(D) \} b^{2k} + \int_{K-(C \cup D)} f^{2k}(x)dx ,
\]

so that

\[
\frac{1}{\sqrt{5}} + \mu(C) + \mu(D) = -\int_{K-(C \cup D)} \left[ \frac{f(x)}{b} \right]^{2k} dx .
\]

Reasoning as before, we see by dominated convergence that the right-hand integral approaches 0 as \( k \to \infty \). But this contradicts the fact that the left side is strictly positive. This contradiction shows that such an \( f \) does not exist.

We remark that the situation is different for Lucas numbers. For let \( 1 = [0,2] \), \( f(x) = a \) if \( 0 \leq x < 1 \), \( f(x) = b \) if \( 1 \leq x \leq 2 \). Then
However, one can show using the methods above that \( f \) cannot be replaced by a continuous function.

**Editorial Note:** Robert Giuli noted that

\[
\int_{a}^{b} x^{n-1} \sqrt{5} \, dx = F_n ,
\]

although this does not satisfy the proposal. It might be interesting to reconsider the proposal with restrictions on \( f \), such as boundedness, etc.

**HISTORY REPEATS**


Prove or disprove the "identity:"

\[
F_{kn} = F_n \sum_{t=1}^{k+1} (-1)^{(n+1)(t+1)} \binom{k-t}{t-1} L_{n}^{k-2t+1} .
\]

where \( F_n \) and \( L_n \) denote the \( n \) th Fibonacci and Lucas numbers, respectively, and \( \lfloor x \rfloor \) denotes the greatest integer function.

**Solution by Douglas Lind, Stanford University**


*Also solved by Wray Brady and L. Carlitz.*
Solve the Diophantine equation,

\[ x^2 + y^2 + 1 = 3xy \]

*Solution by L. Carlitz, Duke University, Durham, North Carolina.*

The equation

\[ (*) \quad x^2 + y^2 + 1 = 3xy \]

can be written in the form

\[ (ax - 3y)^2 - 5y^2 = -4 \]

where \( a = 2 \). We recall that the general (positive) solution of

\[ x^2 - 5y^2 = -4 \]

is given by

\[
\left( \frac{1 + \sqrt{5}}{2} \right)^{2n+1} = \frac{u_n + v_n \sqrt{5}}{2} \quad (n = 0, 1, 2, \ldots),
\]

so that

\[
\begin{align*}
    u_n &= \frac{1}{2^{2n}} \sum_{r=0}^{n} \binom{2n+1}{2r} 5^r, \\
    v_n &= \frac{1}{2^{2n}} \sum_{r=0}^{n} \binom{2n+1}{2r+1} 5^r.
\end{align*}
\]
On the other hand, the Fibonacci number $F_{n+1}$ satisfies

$$F_{n+1} = \frac{1}{2^n} \sum_{2r \leq n} \left( \frac{n + 1}{2r + 1} \right)^2,$$

so that $v_n = F_{2n+1}$. Moreover,

$$u_n + v_n = 2F_{2n+2},$$

which gives

$$u_n = 2F_{2n+2} - F_{2n+1}.$$

Since

$$y = v_n, \quad 2x - 3y = u_n,$$

it follows that

$$2x = u_n + 3v_n = 2F_{2n+2} + 2F_{2n+1} = 2F_{2n+3},$$

so that $x = F_{2n+3}$. Hence we have the general solution of (*) with $x > y$:

$$x = F_{2n+3}, \quad y = F_{2n+1} \quad (n = 0, 1, 2, \cdots).$$

The solution $x = y = 1$ is evidently obtained by taking $n = -1$.


SUM PROJECT

H-175 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put
\[(1 + z + \frac{1}{2}z^2)^{-n-1} = \sum_{k=0}^{\infty} a(n,k)z^k .\]

Show that

(1) \[a(n,n) = \frac{2 \cdot 5 \cdot 8 \cdots (2n - 1)}{n!}\]

\[\sum_{s=0}^{n} \binom{n - s}{s} \binom{2n - s}{n} \left(-\frac{1}{3}\right)^s = \frac{2 \cdot 5 \cdot 8 \cdots (3n - 1)}{n!}\]

(III) \[\sum_{r=0}^{\infty} \binom{n + r}{r} \binom{2n - r}{n} (-\omega)^r = (\omega^2 - \sqrt{3})^n \frac{2 \cdot 5 \cdot 8 \cdots (3n - 1)}{n!},\]

where \[\omega = \frac{1}{2} \left(-1 - \sqrt{3}\right) .\]

**Solution by the Proposer.**

(I) If \(z = w f(z)\), \(f(0) \neq 0\), where \(f(z)\) is analytic about the origin, then (Polya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Vol. 1, p. 125)

\[z = \sum_{n=1}^{\infty} \frac{w^n}{n!} \left[ \frac{d^{n-1}}{dx^{n-1}} (f(x))^n \right]_{x=0} = \sum_{n=0}^{\infty} \frac{w^{n+1}}{(n+1)!} \left[ \frac{d^n}{dx^n} (f(x))^{n+1} \right]_{x=0} .\]
so that

\[
\left[ \frac{d^n}{dx^n} (f(x))^{n+1} \right]_{x=0} = n! \ a(n,n) .
\]

On the other hand, \( z = wf(z) \) becomes

\[
z(1 - z + \frac{1}{3} z^2) = w ,
\]

which reduces to

\[
(1 - z)^3 = 1 - 3w .
\]

It follows that

\[
z = 1 - (1 - 3w)^{\frac{1}{3}}
\]

\[
= \sum_{n=1}^{\infty} (-1)^n \binom{-\frac{1}{3}}{n} 3^n w^n
\]

\[
= \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{3}}{n} 3^n w^n
\]

\[
= \sum_{n=0}^{\infty} \frac{2 \cdot 5 \cdot 8 \cdots (3n - 1)}{(n + 1)!} w^n .
\]

Comparison with (*) gives

\[
a(n,n) = \frac{2 \cdot 5 \cdot 8 \cdots (3n - 1)}{n!} .
\]
(II). Since

\[(1 - z + \frac{1}{3}z^2)^{-n-1} = \sum_{r=0}^{\infty} \binom{n+r}{r} z^r (1 - \frac{1}{3}z)^r \]

\[= \sum_{r=0}^{\infty} \binom{n+r}{r} z^r \sum_{s=0}^{r} \binom{r}{s} \left(-\frac{1}{3}\right)^s z^s \]

\[= \sum_{k=0}^{\infty} z^k \sum_{r+s=k} \binom{n+r}{r} \binom{r}{s} \left(-\frac{1}{3}\right)^s , \]

it follows that

\[a(n,n) = \sum_{s=0}^{n} \binom{n-s}{s} \binom{2n-s}{n} \left(-\frac{1}{3}\right)^s . \]

(III). Put

\[1 - z + \frac{1}{3}z^2 = (1 - \alpha z)(1 - \beta z) . \]

It is easily verified that

\[\alpha = -\frac{\omega^2}{\sqrt{-3}} , \quad \beta = \frac{\omega}{\sqrt{-3}} . \]

Then

\[(1 - z + \frac{1}{3}z^2)^{-n-1} = (1 - \alpha z)^{-n-1}(1 - \beta z)^{-n-1} \]

\[= \sum_{r=0}^{\infty} \binom{n+r}{r} \alpha^r z^r \sum_{s=0}^{\infty} \binom{n+s}{s} \beta^s z^s , \]
so that

\[
\begin{align*}
\sum_{r+s=n} \binom{n+r}{r} \binom{n+s}{s} \rho^r \beta^s \\
= \sum_{r=0}^{\infty} \binom{n+r}{r} \binom{2n-r}{n} \left( -\frac{\omega^2}{\sqrt{-3}} \right)^r \left( \frac{\omega}{\sqrt{-3}} \right)^{n-r} \\
= \frac{\omega^n}{(\sqrt{-3})^n} \sum_{r=0}^{n} \binom{n+r}{r} \binom{2n-r}{n} (-\omega)^r .
\end{align*}
\]

[Continued from page 496.]

**GENERALIZED BASES FOR REAL NUMBERS**


**CHALLENGE**

"In what way does the cubic congruence

\[ x^3 - 15x + 25 \equiv 0 \pmod{p} \]

relate to the Fibonacci numbers?

Generalize to other recurring series."

John Brillhart and Emma Lehmer