A POLYNOMIAL REPRESENTATION OF FIBONACCI NUMBERS

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Douglas Lind has proposed the problem, "Does there exist a polynomial $P(x)$ for which $P(F_k) = F_{nk}$?". In particular, it is known that, if $L_n(x)$ is the n^{th} Lucas Polynomial, then

$$L_n(L_{2k+1}) = L_{(2k+1)n} .$$

The answer to the Fibonacci formulation of the problem is in the affirmative also. The following theorem expresses an infinite number of polynomials, defined recursively, satisfying the properties for Fibonacci numbers analogous to those of the Lucas numbers and the Lucas polynomials.

Theorem.

$$F_{(2n+1)k} = 5^n F_k^{2n+1} - \sum_{n=1}^{n} \binom{2n+1}{n+1-s} \left[(-1)^{k+1}\right]^{n+1-s} F_{(2s-1)k}$$

Proof. Using the Binet form,

$$F_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} ,$$

where

$$\alpha = \frac{1 + \sqrt{5}}{2}$$

and

$$\beta = \frac{1 - \sqrt{5}}{2} ,$$

then

$$F_k^{2n+1} = \left[\frac{\alpha^k - \beta^k}{\alpha - \beta}\right]^{2n+1} = \sum_{s=0}^{2n+1} \binom{2n+1}{s} \frac{(\alpha^k)^{2n+1-s}(-\beta^k)^s}{(\alpha - \beta)^{2n+1}}$$

$$= \left\{\sum_{s=0}^{n} + \sum_{s=n}^{2n+1}\right\} \binom{2n+1}{s} \frac{(\alpha^k)^{2n+1-s}(-\beta^k)^s}{(\alpha - \beta)^{2n+1}} .$$

Noting that

$$\binom{2n+1}{s} = \binom{2n+1}{2n+1-s}$$

and substituting $t = 2n + 1 - s$ in the second of the summations from $s = n$ to $s = 2n + 1$, then

$$F_k^{2n+1} = \sum_{s=0}^{n} \frac{\binom{2n+1}{s}(\alpha\beta)^{ks}(-1)^s}{(\alpha - \beta)^{2n}} \frac{\left[(\alpha^k)^{2n+1-2s} - (\beta^k)^{2n+1-2s}\right]}{(\alpha - \beta)} .$$

Substituting $(\alpha - \beta) = \sqrt{5}$ and $\alpha\beta = -1$, noting the Binet form in the last expression of each term of the summation, and solving for the term in which $s = 0$, the theorem can be obtained.

Examples. For $n = 1$,

$$F_{3k} = 5F_k^3 + (-1)^k 3F_k .$$

Hence, the polynomial $P(x) = 5x^2 + 3x$ satisfies $P(F_{2k}) = F_{6k}$, and the polynomial $P(x) = 5x^3 - 3x$ satisfies $P(F_{2k+1}) = F_{6k+3}$. To determine F_{5k} in terms of F_k by the theorem above, it is necessary to substitute the F_{3k} expression above into the theorem with $n = 2$; one obtains

$$F_{5k} = 25F_k^5 + 25(-1)^k F_k^3 + 5F_k ,$$

so that the polynomials

$$P_1(x) = 25x^5 + 25x^3 + 5x$$

and

$$P_2(x) = 25x^5 - 25x^3 + 5x$$

satisfy

$$P_1(F_{2k}) = F_{10k}$$

and

$$P_2(F_{2k+1}) = F_{10k+5} .$$

Similarly, one may obtain

$$F_{7k} = 5^3 F_k^7 + 7 \cdot 5^2(-1)^k F_k^5 + 70 F_k^3 + 7(-1)^k F_k$$

with polynomials

$$P_3(x) = 125x^7 + 175x^5 + 70x^3 + 7x$$

or

$$P_4(x) = 125x^7 - 175x^5 + 70x^3 - 7x ,$$

where

$$P_3(F_{2k}) = F_{14k}$$

and

$$P_4(F_{2k+1}) = F_{14k+7} .$$

An interesting congruence result may be obtained by taking $2n + 1 =$ odd prime p in the theorem.

$$F_{pk} \equiv 5^{\frac{p-1}{2}} F_k \pmod{p} ,$$

for $(p,5) = 1$. Here,

$$5^{\frac{p-1}{2}} \equiv \left(\frac{5}{p}\right) \pmod{p} ,$$

using the Legendre symbol, and hence

$$F_{p^s k} \equiv F_k \pmod{p}$$

or

$$F_{p^s k} \equiv (-1)^s F_k \pmod{p}$$

according as $p \equiv \pm 1 \pmod 5$ or $p \equiv \pm 2 \pmod 5$.

I would like to take this opportunity to express my sincere appreciation to V. C. Harris for many years of kindness, and years of encouragement and assistance in mathematics. See also [1].

REFERENCE

1. Ellen King, "Some Fibonacci Inverse Trigonometry," unpublished paper, San Jose State College Master's Thesis, 1969.