ON GENERALIZED BASES FOR REAL NUMBERS

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1. INTRODUCTION

The purpose of this paper is to give an exposition of certain results due to J. A. Fridy [1], [2], using a somewhat different approach. In [2], Fridy considers a non-increasing sequence

$$\{r_i\}_1^\infty$$

of real numbers with

$$\lim_{i \to \infty} r_i = 0$$

and defines, for two given non-negative integer sequences

$$\{k_i\}_1^\infty$$

and

$$\{m_i\}_1^\infty ,$$

the sequence $\{r_i\}$ to be a $\{k,m\}$ base for the interval $(-S^*, S)$ if for each $x \in (-S^*, S)$, there is an integer sequence

$$\{a_i\}_1^\infty$$

such that

$$x = \sum_1^\infty a_i r_i$$

with $-m_i \le a_i \le k_i$ for each $i \ge 1$, where

$$S = \sum_1^\infty k_i r_i$$

and

$$S^* = \sum_1^\infty m_i r_i \quad .$$

When the $\{k_i\}$ and $\{m_i\}$ sequences are specialized to $k_i = n - 1$ for all $i \ge 1$ and $m_i = 0$ for all $i \ge 1$, Fridy [1] has termed the resulting $\{k,m\}$ base an "n-base" and developed a necessary and sufficient condition for a sequence $\{r_i\}$ to be an n-base. He also notes in a subsequent paper [2] that a necessary and sufficient condition for a 2-base had been given by Kakeya [3] much earlier. The main result of Fridy's second paper derives from a Lemma which gives a necessary and sufficient condition for $\{r_i\}$ to be a $\{k,0\}$ base ([2], pp. 194-196). Since an n-base is a specialization of a $\{k,0\}$ base, this latter condition for a $\{k,0\}$ base subsumes the earlier result for an n-base in [1]. Moreover, the derivation of the necessary and sufficient condition for a $\{k,m\}$ base follows directly ([2], Theorem 1, pp. 196-197) once the condition for a $\{k,0\}$ base is established.

Our point of departure here is to show that the characterizing condition for a $\{k,0\}$ base is itself almost immediate from Kakeya's condition for a 2-base. This follows from the observation that $\{r_i\}$ is a $\{k,0\}$ base if and only if a certain augmented sequence (obtained by repeating each r_i, in order k_i times) is a 2-base; the details are given below in Theorem 1. (cf. the development in [4].)

In order to keep the presentation self-contained, a proof of Kakeya's result is also given as Lemma 1, where we have emphasized the possibility of obtaining expansions of the required form with an infinite number of the expansion coefficients being equal to zero. This particular constraint will be seen to be important in Section 3, which deals with uniqueness of the expansions.

As illustrations of some of the results, we show in Section 4 that the Cantor expansion is a special case in which unique expansions are obtained. A Lemma is then established which gives a useful sufficient condition for the existence of expansions (non-unique, in general), and this Lemma is applied to show that an arbitrary positive number may be expressed (non-uniquely) as a sum of distinct reciprocal primes. A similar result holds for the Fibonacci numbers

$$\{F_i\}_1^{\infty}$$

where $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \ge 2$; that is, any real number

$$x \in \left(0, \sum_1^{\infty} \frac{1}{F_i}\right)$$

may be represented (again, non-uniquely) as a distinct sum of reciprocal Fibonacci numbers. Along the same lines, we show that any real number

$$x \in \left(-\sum_1^{\infty} \frac{1}{F_i}, \sum_1^{\infty} \frac{1}{F_i}\right)$$

has an expansion of the form

$$x = \sum_1^{\infty} \frac{\epsilon_i}{F_i} ,$$

where each $\epsilon_i = \epsilon_i(x)$ is either a +1 or -1.

2. EXISTENCE OF REPRESENTATIONS

Lemma 1: (KAKEYA): Let

$$\{r_i\}_1^\infty$$

be a non-increasing sequence of real numbers such that

$$\lim_{i \to \infty} r_i = 0$$

and

(1) $$r_p \le \sum_{p+1}^{\infty} r_i \quad \text{for} \quad p = 1, 2, 3, \cdots .$$

If

$$\sum_1^\infty r_i = S ,$$

finite or infinite, then for each x is $[0,S)$, there exist binary coefficients $\alpha_i = \alpha_i(x)$ such that

(2) $$x = \sum_1^\infty \alpha_i r_i$$

and $\alpha_i = 0$ for infinitely many values of i.

Proof. The case $S = +\infty$ is straightforward and left to the reader. It is also apparent that the Lemma holds for $x = 0$.

Now, for S finite, let x be given in $(0,S)$. Choose n_1 as the smallest positive integer such that $r_{n_1} \le x$. If equality holds, the lemma is proved for x; if not, choose n_2 as the smallest integer $>n_1$ for which

$$r_{n_2} \le x - r_{n_1} .$$

Again, equality at this stage implies the result. Otherwise, we continue the process, and in general, n_k is the smallest integer $> n_{k-1}$ for which

$$r_{n_k} \le x - \sum_1^{k-1} r_{n_i} \quad .$$

The process either terminates with an equality sign after a finite number of steps, or else we obtain an infinite series

$$\sum_1^{\infty} r_{n_i} \quad ;$$

we focus our attention on the latter case. Clearly,

$$\sum_1^{\infty} r_{n_i}$$

converges since

$$\sum_1^{p} r_{n_i} \le x$$

for any choice of p. Let

$$\beta = \sum_1^{\infty} r_{n_i} \quad .$$

First, we show $n_i > n_{i-1} + 1$ for infinitely many values of i. If not, there exists a smallest integer k such that $n_{k+j} = n_k + j$ for $j = 1, 2, \cdots$. Then $n_k > 1$, since

$$\beta \le x < \sum_1^\infty r_i = S .$$

If $k = 1$,

$$x \ge \beta = \sum_{n_1}^\infty r_i \ge r_{n_1 - 1} ,$$

thereby contradicting our choice of n_1. Hence, $k > 1$, and we write

$$\beta = \sum_1^{k-1} r_{n_i} + \sum_{n_k}^\infty r_i$$

with $n_k > n_{k-1} + 1$ from our definition of k. Then

$$x - \sum_1^{k-1} r_{n_i} \ge \beta - \sum_1^{k-1} r_{n_i} = \sum_{n_k}^\infty r_i \ge r_{n_k - 1} ,$$

which implies $n_k = n_{k-1} + 1$, a contradiction. We conclude $n_i > n_{i-1} + 1$ for infinitely many i.

Lastly, we show $\beta = x$. For, if not, $\beta < x$ and there exists N such that $p \ge N$ implies

$$r_{n_p} < x - \beta = x - \sum_1^\infty r_{n_i} \le x - \sum_1^p r_{n_i} ,$$

which in turn implies $n_{p+1} = n_p + 1$ for each $p \ge N$, a contradiction to our previous assertion. q.e.d.

The principal Lemma in Fridy's paper ([2], pp. 194-196) may now be derived quite simply from Lemma 1:

Theorem 1. Let

$$\{r_i\}_1^\infty$$

be a non-increasing sequence of real numbers with $\lim_{i\to\infty} r_i = 0$ and let

$$\{k_i\}_1^\infty$$

be an arbitrary sequence of positive integers. Then every real number x in

$$\left[0, \sum_1^\infty k_i r_i\right)$$

can be expanded in the form

$$x = \sum_1^\infty \beta_i r_i , \tag{3}$$

with β_i integers satisfying $0 \le \beta_i \le k_i$ for $i = 1, 2, \cdots$ if and only if

$$r_p \le \sum_{p+1}^\infty k_i r_i \qquad \text{for} \quad p = 1, 2, 3, \cdots . \tag{4}$$

Further, the expansion in (3) can be accomplished such that $\beta_i < k_i$ for infinitely many values of i.

Proof. To show necessity of (4), assume there exists $m > 0$ such that

$$r_m > \sum_{m+1}^\infty k_i r_i$$

and choose x such that

$$\sum_{m+1}^{\infty} k_i r_i < x < r_m .$$

If x has an expansion of the form (3), we must have $\beta_1 = \beta_2 = \cdots = \beta_m = 0$ since $x < r_m$, but then

$$x = \sum_{m+1}^{\infty} \beta_i r_i \le \sum_{m+1}^{\infty} k_i r_i < x ,$$

a contradiction.

Conversely, assume (4) holds and consider the sequence

$$\{g_i\}_1^{\infty} ,$$

defined to consist of each term r_i, in order, repeated k_i times; that is

$$\{g_i\}_1^{\infty} = \underbrace{r_1, r_1, r_1,}_{k_1 \text{ times}} \quad \underbrace{r_2, r_2, r_2, r_2,}_{k_2 \text{ times}} \cdots , \underbrace{r_n, r_n, r_n,}_{k_n \text{ times}} \cdots .$$

Using (4), we observe

$$g_p \le \sum_{p+1}^{\infty} g_i$$

for $p = 1, 2, 3, \cdots$. Thus, Lemma 1 guarantees binary coefficients α_i such that any x in

$$\left[0, \sum_{1}^{\infty} g_i\right)$$

has an expansion of the form

$$x = \sum_{1}^{\infty} \alpha_i g_i \tag{5}$$

with $\alpha_i = 0$ for infinitely many i. Replacing (5) in terms of the r_i, and noting

$$\sum_{1}^{\infty} g_i = \sum_{1}^{\infty} k_i r_i ,$$

we have that any x in

$$\left[0, \sum_{1}^{\infty} k_i r_i\right)$$

can be written in the form

$$x = \sum_{1}^{\infty} \beta_i r_i$$

with $0 \le \beta_i \le k_i$ and $\beta_i < k_i$ for infinitely many i. q. e. d.

3. UNIQUENESS OF REPRESENTATIONS

Thus, condition (4) is both necessary and sufficient for the existence of expansions in the form (3). We give a result next in Lemma 2 concerning the uniqueness of such expansions independently of the existence question.

Definition. Let

$$\{r_i\}_1^{\infty}$$

be a non-increasing sequence of real numbers with $\lim_{i \to 0} r_i = 0$ and let

$$\{k_i\}_1^\infty$$

be an arbitrary but fixed sequence of positive integers. Let

$$\{\beta_i\}_1^\infty \quad \text{and} \quad \{\gamma_i\}_1^\infty$$

be two sequences of integers which satisfy $0 \le \beta_i \le k_i$ and $0 \le \gamma_i \le k_i$ for $i = 1, 2, 3, \cdots$. Further, let $\beta_i < k_i$ for infinitely many i and $\gamma_i < k_i$ for infinitely many i. Then

$$\{r_i\}_1^\infty$$

will be said to possess the uniqueness property [Property U] if and only if the equality

$$\sum_1^\infty \beta_i r_i = \sum_1^\infty \gamma_i r_i$$

implies $\beta_i = \gamma_i$ for each $i \ge 1$.

Lemma 2. Let

$$\{r_i\}_1^\infty \quad \text{and} \quad \{k_i\}_1^\infty$$

be given as in the preceding definition. Then

$$\{r_i\}_1^\infty$$

possesses Property U if

$$r_p \ge \sum_{p+1}^\infty k_i r_i \quad \text{for} \quad p = 1, 2, 3, \cdots . \tag{6}$$

Proof. Assume (6) holds and that

$$\sum_{1}^{\infty} \beta_i r_i = \sum_{1}^{\infty} \gamma_i r_i$$

with $\{\beta_i\}$ and $\{\gamma_i\}$ as in the definition. If the two representatives are not identical, let m be the smallest positive integer i such that $\beta_i \neq \gamma_i$. Then

$$\beta_m r_m + \sum_{m+1}^{\infty} \beta_i r_i = \gamma_m r_m + \sum_{m+1}^{\infty} \gamma_i r_i ,$$

or assuming $\beta_m > \gamma_m$ without loss of generality,

$$(\beta_m - \gamma_m) = \sum_{m+1}^{\infty} (\gamma_i - \beta_i) r_i . \tag{7}$$

Now, $\gamma_i - \beta_i < k_i$ for some $i \geq m + 1$ (otherwise $\gamma_i = k_i$ for all $i \geq m + 1$, contrary to choice of $\{\gamma_i\}$), and therefore, from (7),

$$r_m \leq (\beta_m - \gamma_m) r_m < \sum_{m+1}^{\infty} k_i r_i ,$$

contradicting condition (6) for $p = m$. We conclude $\gamma_i = \beta_i$ for all $i \geq 1$, giving Property U. q. e. d.

Lemma 3. Take

$$\{r_i\}_1^{\infty} \quad \text{and} \quad \{k_i\}_1^{\infty}$$

as before. If

$$r_p \le \sum_{p+1}^{\infty} k_i r_i$$

for $p = 1, 2, 3, \cdots$, then

(8) $$r_p = \sum_{p+1}^{\infty} k_i r_i \qquad (p = 1, 2, 3, \cdots)$$

is a necessary and sufficient condition for $\{r_i\}$ to possess Property U.

Proof. Sufficiency follows from Lemma 2. To show necessity, assume that there exists an integer $m > 0$ such that

$$r_m < \sum_{m+1}^{\infty} k_i r_i ,$$

and choose x to satisfy

$$r_m < x < \sum_{m+1}^{\infty} k_i r_i .$$

By Theorem 1, x has an expansion of the form

$$x = \sum_{1}^{\infty} \beta_i r_i$$

with $0 \le \beta_i \le k_i$ for $i \ge 1$ and $\beta_i < k_i$ for many i. Further, at least one of the coefficients $\beta_1, \beta_2, \cdots, \beta_m$ must be different from zero.

Since the sequence

$$\{r_i\}_{m+1}^{\infty}$$

also satisfies the conditions of Theorem 1 and

$$x < \sum_{m+1}^{\infty} k_i r_i ,$$

the number x has an expansion of the form

$$x = \sum_{m+1}^{\infty} \gamma_i r_i$$

with $0 \le \gamma_i \le k_i$ for $i \ge m + 1$ and $\gamma_i < k_i$ for infinitely many i. Thus

$$x = \sum_{m+1}^{\infty} \gamma_i r_i = \sum_{1}^{\infty} \beta_i r_i$$

and $\beta_i = \gamma_i$ does not hold for all $i \ge 1$, showing Property U does not hold. q. e. d.

Theorem 2. Let

$$\{r_i\}_1^{\infty} \quad \text{and} \quad \{k_i\}_1^{\infty}$$

be sequences as in Theorem 1. Then every real number x in

$$\left[0, \sum_{1}^{\infty} k_i r_i\right)$$

has one and only one expansion

(8)
$$x = \sum_{1}^{\infty} \beta_i r_i$$

with $0 \le \beta_i \le k_i$ for $i \ge 1$ and $\beta_i < k_i$ for infinitely many i, if and only if

(9)
$$r_p = \sum_{p+1}^{\infty} k_i r_i$$

for $p = 1, 2, 3, \cdots$, or equivalently,

(10)
$$r_p = S \cdot \prod_{i=1}^{p} \frac{1}{1 + k_i}$$

for all $p \ge 1$, where

$$S = \sum_{1}^{\infty} k_i r_i .$$

<u>Proof</u>. From Theorem 1, we must have

$$r_p \le \sum_{p+1}^{\infty} k_i r_i$$

for $p \ge 1$, while from Lemma 3 and the uniqueness requirement,

$$r_p = \sum_{p+1}^{\infty} k_i r_i$$

for $p \geq 1$. Equation (10) follows on noting

$$r_{p+1} = \sum_{p+2}^{\infty} k_i r_i = r_p - k_{p+1} r_{p+1} ,$$

or

(11) $$r_{p+1} = \frac{r_p}{1 + k_{p+1}}$$

for $p \geq 1$. Since

$$r_1 = \sum_{2}^{\infty} k_i r_i = S - r_1 k_1,$$

we have

$$r_1 = \frac{S}{1 + k_1} ,$$

and iteration using (11) leads to (10). q.e.d.

4. APPLICATIONS

CANTOR EXPANSION ([5], Theorem 1.6, p. 7): "Let $a_1, a_2, a_3, \cdots$ be a sequence of positive integers, all greater than 1. Then any real number α is uniquely expressible in the form

(12) $$\alpha = c_0 + \sum_{i=1}^{\infty} \frac{c_i}{a_1 a_2 \cdots a_i}$$

with integers c_i satisfying the inequalities $0 \leq c_i \leq a_i - 1$ for all $i \geq 1$ and $c_i < a_i - 1$ for infinitely many i."

Proof. In Theorem 2, identify

$$r_i = \frac{1}{a_1 a_2 \cdots a_i}$$

and $k_i = a_i - 1$ for $i \geq 1$. Then condition (11) is clearly satisfied. Now, for given α, let $c_0 = [\alpha]$, the greatest integer contained in α, so that

$$0 \leq \alpha - [\alpha] < 1 = \sum_1^\infty k_i r_i = \sum_1^\infty \frac{a_i - 1}{a_1 a_2 \cdots a_i} .$$

Then Theorem 2 implies a unique expansion in the form (12) as required. q. e. d.

Next, we give a useful sufficient condition for the existence of expansions as specified in Theorem 1.

Lemma 4. A sufficient condition for

$$r_p \leq \sum_{p+1}^\infty k_i r_i \qquad (p \geq 1)$$

is

$$r_p \leq (k_{p+1} + 1) r_{p+1} \tag{13}$$

for all $p \geq 1$.

Proof. Assume (13) is satisfied. Then

$$\sum_{p+1}^\infty r_i \leq \sum_{p+1}^\infty (k_{i+1} + 1) r_{i+1} = \sum_{p+1}^\infty k_{i+1} r_{i+1} + \sum_{p+1}^\infty r_{i+1} .$$

Thus,

$$r_{p+1} = \sum_{p+1}^{\infty} r_i - \sum_{p+1}^{\infty} r_{i+1} \le \sum_{p+1}^{\infty} k_{i+1} r_{i+1} = \sum_{p+1}^{\infty} k_i r_i - k_{p+1} r_{p+1}$$

or

$$(1 + k_{p+1}) r_{p+1} \le \sum_{p+1}^{\infty} k_i r_i .$$

Since $r_p \le (1 + k_{p+1}) r_{p+1}$, we have

$$r_p \le \sum_{p+1}^{\infty} k_i r_i$$

for all $p \ge 1$ as required.

Example 1. Let x be an arbitrary real number satisfying

$$0 \le x < \sum_{1}^{\infty} \frac{1}{F_i} ,$$

where $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n \ge 2$ specify the Fibonacci numbers. Then

$$x = \sum_{1}^{\infty} \frac{\alpha_i}{F_i} ,$$

with $\alpha_i = \alpha_i(x)$ a binary coefficient for each $i \ge 1$. Further, $\alpha_i = 0$ for infinitely many i.

Proof. Here $k_i = 1$ for all $i \ge 1$. Clearly

$$\left\{\frac{1}{F_i}\right\}_1^\infty$$

is non-increasing and

$$\lim_{i\to\infty}\frac{1}{F_i} = 0 .$$

By condition (13) of Lemma 4, a sufficient condition for Theorem 1 to be applicable is $r_p \le 2r_{p+1}$, or equivalently,

$$\frac{1}{F_p} \le \frac{2}{F_{p+1}} \qquad (p \ge 1) ,$$

where

$$\{r_i\}_1^\infty = \left\{\frac{1}{F_i}\right\}_1^\infty .$$

But this is merely the condition $F_{p+1} \le 2F_p$, which is obvious for $p \ge 1$ and the result follows from Theorem 1.

Example 2. Let x be an arbitrary real number satisfying $0 \le x < \infty$. Then

$$x = \sum_1^\infty \frac{\alpha_i}{p_i} ,$$

where

$$\{p_i\}_1^\infty = \{2, 3, 5, 7, 11, \cdots\}$$

is the sequence of primes and $\alpha_i = \alpha_i(x)$ is a binary coefficient for each $i \geq 1$. Further $\alpha_i = 0$ for infinitely many i.

Proof. Again, we apply Theorem 1 with

$$r_i = \frac{1}{p_i}$$

for $i \geq 1$ and $k_i = 1$ for all $i \geq 1$. Condition (13) reduces to $p_{i+1} \leq 2p_i$, and this latter inequality holds for all $i \geq 1$ by Betrand's postulate ([6], p. 171). Since

$$\left\{\frac{1}{p_i}\right\}_1^\infty$$

is non-increasing and

$$\lim_{i\to\infty} \frac{1}{p_i} = 0,$$

the result follows from Theorem 1 and the well-known divergence of the series

$$\sum_1^\infty \frac{1}{p_i}$$

([6], Theorem 8.3, p. 168).

Example 3. Let x be an arbitrary real number with

$$-\sum_1^\infty \frac{1}{F_i^2} \leq x \leq \sum_1^\infty \frac{1}{F_i^2} .$$

Then x possesses an expansion of the form

$$x = \sum_1^\infty \frac{\epsilon_i}{F_i} , \tag{14}$$

where each $\epsilon_i = \epsilon_i(x)$ is either +1 or -1.

Proof. For

$$x \in \left(-\sum_1^\infty \frac{1}{F_i}, \quad \sum_1^\infty \frac{1}{F_i} \right),$$

we have

$$0 < \frac{1}{2}\left(x + \sum_1^\infty \frac{1}{F_i} \right) < \sum_1^\infty \frac{1}{F_i} \quad ,$$

so that by Example 1,

$$\frac{1}{2}\left(x + \sum_1^\infty \frac{1}{F_i} \right) = \sum_1^\infty \frac{\alpha_i}{F_i} \quad ,$$

where each α_i is a binary digit. Equivalently,

$$x = \sum_1^\infty \frac{2\alpha_i - 1}{F_i} \quad ,$$

and we note that $2\alpha_i - 1$ is either +1 or -1 depending on whether $\alpha_i = 1$ or $\alpha_i = 0$, respectively; this establishes the expansion in the stated form.

REFERENCES

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2. J. A. Fridy, "Generalized Bases for the Real Numbers," The Fibonacci Quarterly, Vol. 4, No. 3, October, 1966, pp. 193-201.

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