1. INTRODUCTION

In a recent paper [1], Brother Alfred Brousseau has obtained a chain of formulas of the following kind.

\[
\sum_{n=1}^{\infty} \frac{1}{F_n} = 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+2}}
\]

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+2}} = \frac{5}{12} - \frac{3}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}}
\]

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} \cdots F_{n+4}} = \frac{97}{2640} + \frac{40}{41} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{F_n F_{n+1} \cdots F_{n+6}}
\]

As an application he has computed the value of the sum

\[
S = \sum_{n=1}^{\infty} \frac{1}{F_n}
\]

to twenty-five decimal places. It does not seem to be known whether the sum \( S \) is a rational number.

If we define

\[
S_k = \sum_{n=1}^{\infty} \frac{(-1)^{k(n-1)}}{F_n F_{n+1} \cdots F_{n+2k}} \quad (k = 0, 1, 2, \cdots)
\]

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the above special results suggest that generally

\[ S_{k+1} = a_k + b_k S_k, \]

where \( a_k \) and \( b_k \) are rational numbers. We shall show below that this is indeed true and moreover we shall obtain explicit formulas for \( a_k \) and \( b_k \). Also we obtain explicit formulas for the sum

\[ \sum_{n=1}^{\infty} \frac{(-1)^{k(n-1)}}{F_n F_{n+1} \cdots F_{n+2k-1}} \quad (k = 1, 2, 3, \cdots). \]

Indeed we shall prove these results in a somewhat more general setting. In place of the Fibonacci numbers \( F_n \) we take the numbers \( u_n \) defined by

\[ u_0 = 0, \quad u_1 = 1, \quad u_{n+1} = (\alpha + \beta)u_n - \alpha \beta u_{n-1} \quad (n = 1, 2, 3, \cdots), \]

where \( \alpha, \beta \) are distinct, and consider the sums

\[ U_k = \sum_{n=1}^{\infty} \frac{(\alpha \beta)^{nk}}{u_n u_{n+1} \cdots u_{n+2k}} \]

and

\[ T_k = \sum_{n=1}^{\infty} \frac{(\alpha \beta)^{nk}}{u_n u_{n+1} \cdots u_{n+2k-1}}. \]

We show that

\[ U_{k+1} = c_k + d_k U_k, \]

where \( c_k \) and \( d_k \) are rational functions of \( \alpha, \beta \) that are determined explicitly. As for \( T_k \), we show that

\[ T_k = c_k' + \frac{d_k'}{\alpha}, \]

where \( c_k' \) and \( d_k' \) are rational functions of \( \alpha, \beta \) that are determined explicitly. Also it is assumed, in order to assure convergence, that
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| α | > | β |, | α | > 1.

2. SOME PRELIMINARY RESULTS

To begin with, let \( \alpha, \beta \) denote indeterminates and put

\[
(2.1) \quad u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad v_n = \alpha^n + \beta^n.
\]

Then, of course,

\[
(2.2) \quad \begin{cases} u_{n+1} = (\alpha + \beta)u_n - \alpha\beta u_{n-1} \\ v_{n+1} = (\alpha + \beta)v_n - \alpha\beta v_{n} \end{cases}.
\]

Next define

\[
(2.3) \quad (u)_0 = 1, \quad (u)_n = u_1u_2 \cdots u_n
\]

and

\[
(2.4) \quad \binom{n}{k} = \frac{(u)_n}{(u)_k(u)_{n-k}} = \binom{n}{n-k}.
\]

It follows from the definition that

\[
(2.5) \quad \begin{cases} n + 1 \choose k \\ k + n \choose k \end{cases} = \alpha^k \binom{n}{k} + \beta^{n-k+1} \binom{n}{k-1} = \beta^k \binom{n}{k} + \alpha^{n-k+1} \binom{n}{k-1}.
\]

Clearly \( u_n, v_n, \binom{n}{k} \) are symmetric polynomials in \( \alpha, \beta \); the last assertion is a consequence of (2.5).

Let
(2.6) \[ R_{2k}(x) = \sum_{j=0}^{2k} (-1)^j \left\{ \binom{2k}{j} + \binom{2k}{j+1} \right\} (\alpha x)^j (\beta x)^{j+1}, \]

(2.7) \[ R_{2k-1}(x) = \sum_{j=0}^{2k-1} (-1)^j \left\{ \binom{2k}{j} - \binom{2k}{j+1} \right\} (\alpha x)^j (\beta x)^{j+1}. \]

Then, by (2.5),

\[
R_{2k}(x) = \sum_{j=0}^{2k} (-1)^j \left\{ \binom{2k}{j} + \binom{2k}{j+1} \right\} (\alpha x)^j (\beta x)^{j+1} - \sum_{j=0}^{2k-1} (-1)^j \left\{ \binom{2k}{j} - \binom{2k}{j+1} \right\} (\alpha x)^j (\beta x)^{j+1}.
\]

Similarly,

(2.8) \[ R_{2k}(x) = (\alpha^{-1} - \alpha^{-k} \beta^k x) R_{2k-1}(x). \]
\[ R_{2k+1}(x) = \sum_{j=0}^{2k+1} (-1)^j \left\{ \frac{2k + 1}{j} \right\} \left[ \alpha^j (j+1)(j+2) - j(k+1) \right] \frac{1}{x^j} \]

\[ = \sum_{j=0}^{2k+1} (-1)^j \frac{1}{\alpha^j} (j+1)(j+2) - j(k+1) \frac{1}{\beta^j} j(j+1) - j(k+1) \frac{1}{x^j} \]

\[ = \left[ \frac{\beta^j}{2} \right] \left\{ \frac{2k + 1}{j} \right\} + 2k - j + 1 \left\{ \frac{2k}{j - 1} \right\} \]

\[ = \sum_{j=0}^{2k} (-1)^j \left\{ \frac{2k}{j} \right\} \left[ \alpha^j (j+1)(j+2) - jk - j \beta^j j(j+1) - j \right] \]

\[ = \sum_{j=0}^{2k} (-1)^j \left\{ \frac{2k}{j} \right\} \left[ \alpha^j (j+1)(j+2) - jk - j \beta^j j(j+1) - j \right] \]

\[ = (\alpha - \alpha^{k+1} \beta^{-k}) \sum_{s=0}^{2k} (-1)^k \left\{ \frac{2k}{j} \right\} \left[ \alpha^j (j+1)(j+2) - jk - j \beta^j j(j+1) - j \right] \]

and so

\[ (2.9) \quad R_{2k+1}(x) = (\alpha - \alpha^{k+1} \beta^{-k}) R_{2k}(x). \]

Combining (2.8) and (2.9) we get

\[ (2.10) \quad R_{2k}(x) = (\alpha \beta)^{-k} (\alpha^{k-1} - \beta^{k+1}) (\beta^{k-1} - \alpha^{k}) R_{2k-2}(x) \quad (k \geq 1) \]

and therefore

\[ (2.11) \quad R_{2k}(x) = (\alpha \beta)^{-\frac{k}{2}} (k-1) \left[ \prod_{j=1}^{k} (\alpha^j - \beta^j)(\beta^{j-1} - \alpha^j) \right] \]

\[ = (\alpha \beta)^{-\frac{k}{2}} (k-1) \left[ \prod_{j=1}^{k} (\alpha \beta)^{j-1} - \beta^{2j-1} + (\alpha \beta)^{j} \beta^2 \right] . \]
with \( v_{2j-1} \) defined by (2.1).

The recurrence (2.10) can be generalized in the following way. Let

\[
\xi = (x_0, x_1, x_2, \cdots)
\]

denote an arbitrary sequence and define

\[
R_{2k}(\xi) = \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (\alpha \beta)^{j+1} - jk \cdot x_j.
\]

Then, exactly as above, we have

\[
R_{2k}(\xi) = (\alpha \beta)^{-k+1} \sum_{j=0}^{2k-2} (-1)^j \binom{2k}{j} (\alpha \beta)^{j+1} - j(k-1)
\]

\[
\cdot \left[ (\alpha \beta)^{k-1} x_j - v_{2k-1} x_{j+1} + (\alpha \beta)^k x_{j+2} \right].
\]

It follows from (2.12) that

\[
R_{2k+2}(\xi) = R_{2k}(\xi)
\]

\[
= \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (\alpha \beta)^{j+1} - jk \left[ (\alpha \beta)^{-k} v_{2k+1} x_{j+1} + \alpha \beta x_{j+2} \right]
\]

3. A SECOND PROOF OF EQ. (2.11)

It may be of interest to show that (2.11) can be obtained from a known result. We recall that
The reduction formulas for Fibonacci summations are given by

\[(3.1) \quad \prod_{j=0}^{k-1} (1 - q^j x) = \sum_{j=0}^{k} (-1)^j \left[ \binom{k}{j} q^j (1-q)^{j-1} x^j \right] , \]

where

\[ \left[ \binom{k}{j} \right] = \frac{(1 - q^k)(1 - q^{k-1}) \cdots (1 - q^{j+1})}{(1 - q)(1 - q^2) \cdots (1 - q^j)} . \]

Replacing \( q \) by \( \alpha/\beta \),

\[ \left[ \binom{k}{j} \right] \rightarrow \left( \frac{\beta^k - \alpha^k}{\beta^j - \alpha^j} \right) \frac{(\beta^{k-j} - \alpha^{k-j}) \cdots (\beta^{k-1} - \alpha^{k-1})}{(\beta^j - \alpha^j)(\beta^{j+1} - \alpha^{j+1})} \beta^{j-k} \]

Thus (2.1) becomes

\[ \beta^{-k(2k-1)} \prod_{j=0}^{2k-1} (\beta^j - \alpha^j x) = \sum_{j=0}^{2k} (-1)^j \left\{ \binom{2k}{j} \beta^{j-1} (1-j-k)x^j \right\} . \]

In particular, if \( k \) is replaced by \( 2k \), we get

\[(3.2) \quad \beta^{-k(2k-1)} \prod_{j=0}^{2k-1} (\beta^j - \alpha^j x) = \sum_{j=0}^{2k} (-1)^j \left\{ \binom{2k}{j} \beta^{j-1} (1-j-2k)x^j \right\} . \]

Now replace \( x \) by \( \alpha^{-k} \beta^k x \) and (3.2) becomes

\[ \beta^{-k(2k-1)} \prod_{j=0}^{2k-1} (\beta^j - \alpha^{-k+j+1} \beta^k x) \]

\[ = \sum_{j=0}^{2k} (-1)^j \left\{ \binom{2k}{j} \alpha^j \beta^{j-1} (1-j-1)x^j \right\} . \]
so that

\[ R_{2k}(x) = \beta^{-k(2k-1)} \prod_{j=0}^{2k-1} (\beta^{j} - \alpha^{j-k+1} \beta^{k} x) \]

(3.3)

\[ = \prod_{j=0}^{2k-1} (1 - \alpha^{j-k+1} \beta^{k+j-1} x) = \prod_{j=1}^{2k} (1 - \alpha^{k-j+1} \beta^{j-k} x) ; \]

at the last step we have replaced \( j \) by \( 2k - j \).

Now on the other hand,

\[ \prod_{j=1}^{k} (\alpha^{j-1} - \beta^{j} x)(\beta^{j-1} - \alpha^{j-1} x) \]

\[ = (\alpha \beta)^{k(k-1)} \prod_{k=1}^{k-1} (1 - \alpha^{j+1} \beta^{i} x)(1 - \alpha^{j} \beta^{j+1} x) \]

\[ = (\alpha \beta)^{k(k-1)} \prod_{j=0}^{k-1} (1 - \alpha^{j-k+1} \beta^{k-j} x)(1 - \alpha^{k-j} \beta^{k-j+1} x) \]

\[ = (\alpha \beta)^{k(k-1)} \prod_{j=0}^{k-1} (1 - \alpha^{k-j+1} \beta^{j-k} x) \cdot \prod_{j=k}^{2k-1} (1 - \alpha^{j-k+1} \beta^{j-k} x) \]

\[ = (\alpha \beta)^{k(k-1)} \prod_{j=0}^{2k-1} (1 - \alpha^{j-k+1} \beta^{k-j} x) . \]

Substitution in (3.3) gives

\[ R_{2k}(x) = (\alpha \beta)^{k(k-1)} \prod_{j=1}^{k} (\alpha^{j-1} - \beta^{j} x)(\beta^{j-1} - \alpha^{j} x) , \]

which is the first of (2.11).

4. THE MAIN RESULTS

We consider next the expansion into partial fractions of
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\[ (4.1) \quad \frac{x^k}{(1 - x)(\alpha - \beta x)(\alpha^2 - \beta^2 x) \cdots (\alpha^{2k} - \beta^{2k} x)} = \sum_{j=0}^{2k} \frac{A_j}{\alpha^j - \beta^j x}, \]

where \( A_j \) is independent of \( x \). We find that

\[ (4.2) \quad (\alpha - \beta)^{2k} A_j = (-1)^j \frac{(\alpha \beta)^{\frac{j}{2}(j+1) - jk}}{(u)_{2k-j}}, \]

where, as above,

\[ (u)_n = \prod_{j=1}^{n} \frac{\alpha^j - \beta^j}{\alpha - \beta}. \]

Thus we have the identity

\[ (4.3) \quad \frac{(\alpha - \beta)^{2k} x^k}{(1 - x)(\alpha - \beta x) \cdots (\alpha^{2k} - \beta^{2k} x)} = \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \frac{(\alpha \beta)^{\frac{j}{2}(j+1) - jk}}{(u)_{2k-j}} \]

For \( x = \alpha^{-n} \beta^n \), the left member of (4.3) becomes

\[ \frac{\alpha^{n(k+1)} \beta^{nk}}{\alpha - \beta} \frac{\alpha^n u_{n+1} \cdots u_{n+2k}}{u_n u_{n+1} \cdots u_{n+2k}}, \]

while the right member becomes

\[ \frac{\alpha^n}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \frac{(\alpha \beta)^{\frac{j}{2}(j+1) - jk}}{(u)_{2k-j}} \frac{\alpha^{n(j+1)} - \beta^{n(j+1)}}{\alpha^{n+j} - \beta^{n+j}}. \]
We have therefore the identity

\[ (4.4) \quad \frac{(\alpha \beta)^{nk}}{u_n u_{n+1} \cdots u_{n+2k}} = \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (\alpha \beta)^{j(j+1)-jk} \sum_{n=1}^{\infty} \frac{1}{\alpha^{n+j} - \beta^{n+j}}. \]

Now put

\[ (4.5) \quad U_k = \sum_{n=1}^{\infty} \frac{(\alpha \beta)^{nk}}{u_n u_{n+1} \cdots u_{n+2k}} \quad (k = 0, 1, 2, \cdots) \]

and in particular, for \( k = 0 \),

\[ (4.6) \quad U = U_0 = \sum_{n=1}^{\infty} \frac{1}{u_n}. \]

To assure convergence, we assume that

\[ |\alpha| > |\beta|, \quad |\alpha| = 1. \]

Then, by (4.4) and (4.5),

\[
U = \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (\alpha \beta)^{j(j+1)-jk} \sum_{n=1}^{\infty} \frac{1}{u_{n+j}} \sum_{n=1}^{\infty} \frac{1}{u_n}.
\]

\[
= \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (\alpha \beta)^{j(j+1)-jk} \cdot U
\]

\[
- \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (\alpha \beta)^{j(j+1)-jk} \sum_{n=1}^{\infty} \frac{1}{u_n}.
\]
The coefficient on the right is equal to

\[
\frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (\alpha \beta)^j (j+1) - jk
\]

\[
= \frac{1}{(u)_{2k}} S_{2k}(1)
\]

\[
= (\alpha \beta)^{\frac{1}{2} k (k-1)} \frac{\prod_{j=1}^{k} \left( (\alpha \beta)^{j-1} - v_{2j+1} + (\alpha \beta)^{j} \right)}{(u)_{2k}}
\]

We have therefore

\[
U_k = (\alpha \beta)^{\frac{1}{2} k (k-1)} \frac{\prod_{j=1}^{k} \left( (\alpha \beta)^{j-1} - v_{2j+1} + (\alpha \beta)^{j} \right)}{(u)_{2k}} U \cdot \sum_{n=1}^{j} \frac{1}{u_n}
\]

(4.7)

\[
- \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (\alpha \beta)^j (j+1) - jk \sum_{n=1}^{j} \frac{1}{u_n}
\]

More generally, if we put

(4.8)

\[
U_k(x) = \sum_{n=1}^{\infty} \frac{(\alpha \beta)^{nk} x^{n+2k}}{u_n u_{n+1} \cdots u_{n+2k}}
\]

and in particular, for \( k = 0 \),

(4.9)

\[
U(x) = U_0(x) = \sum_{n=1}^{\infty} \frac{x^n}{u_n}
\]

then as above,
\[ U_k(x) = \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (\alpha \beta)^{j(j+1)-jk} \sum_{n=1}^{\infty} x^{n+2k} u_n \]

\[ = \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (\alpha \beta)^{j(j+1)-jk} x^{2k-j} U(x) \]

\[ - \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (\alpha \beta)^{j(j+1)-jk} x^{2k-j} \sum_{n=1}^{\infty} \frac{x^n}{u_n} . \]

Since

\[ \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (\alpha \beta)^{j(j+1)-jk} x^{2k-j} = x^{2k} S_{2k}(x^{-1}) \]

\[ = (\alpha \beta)^{-\frac{1}{2}} k(1-k) x^{2k} \prod_{j=1}^{k} [(\alpha \beta)^{j-1} - v_{2j-1} x^{-1} + (\alpha \beta)^{j-2}] \]

\[ = (\alpha \beta)^{-\frac{1}{2}} k(1-k) \prod_{j=1}^{k} [(\alpha \beta)^{j-1} x^2 - v_{2j-1} x + (\alpha \beta)^j] \]

it is clear that

\[ U_k(x) = \frac{(\alpha \beta)^{-\frac{1}{2}} k(1-k)}{(u)_{2k}} U(x) \prod_{j=1}^{k} [(\alpha \beta)^{j-1} x^2 - v_{2j-1} x + (\alpha \beta)^j] \]

(4.10)

\[ - \frac{1}{(u)_{2k}} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (\alpha \beta)^{j(j+1)-jk} x^{2k-j} \sum_{n=1}^{\infty} \frac{x^n}{u_n} . \]

It follows from (4.10) that
(4.11) \[ U_{k+1}(x) = \frac{x^2 - (\alpha \beta)^k v_{2k+1} x + \alpha \beta}{u_{2k+1} u_{2k+2}} U_k(x) \]

\[ = -\frac{x^{2k+2}}{(u)_{2k+2}} \left\{ \sigma_{2k+2}(x) - \left[ 1 - (\alpha \beta)^{-k} v_{2k+1} x^{-1} + \alpha \beta x^{-2} \right] \sigma_{2k}(x) \right\}, \]

where

\[ \sigma_{2k}(x) = \sum_{j=0}^{2k} (-1)^j \left\{ \begin{array}{c} 2k \end{array} \right\} (\alpha \beta)^{j(j+1)-jk} x^{-j} \sum_{n=1}^{j} \frac{x^n}{u_n}. \]

If we now apply (2.13) to \( \sigma_{2k}(x) \) with

(4.12) \[ x_j = x^{-j} \sum_{n=1}^{j} \frac{x^n}{u_n}, \]

we get

\[ \sigma_{2k+2}(x) - \sigma_{2k}(x) = \sum_{j=0}^{2k} (-1)^j \left\{ \begin{array}{c} 2k \end{array} \right\} (\alpha \beta)^{j(j+1)-jk} \]

\[ \cdot \left[ -(\alpha \beta)^{-k} v_{2k+1} x^{j+1} + \alpha \beta x^{j+2} \right]. \]

Thus, by (4.12), (4.11) reduces to

(4.13) \[ U_{k+1}(x) = \frac{x^2 - (\alpha \beta)^{-k} v_{2k+1} x + \alpha \beta}{u_{2k+1} u_{2k+2}} U_k(x) \]

\[ = \frac{x^{2k+2}}{(u)_{2k+2}} \sum_{j=0}^{2k} (-1)^j \left\{ \begin{array}{c} 2k \end{array} \right\} (\alpha \beta)^{j(j+1)-jk} \]

\[ \cdot \left[ (\alpha \beta)^{-k} \frac{v_{2k+1}}{u_{j+1}} - \alpha \beta \left( \frac{x^{-1}}{u_{j+1}} + \frac{1}{u_{j+2}} \right) \right]. \]

In particular, for \( x = 1 \), (4.13) becomes
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\[ U_{k+1} = \frac{1 - (\alpha \beta)^{-k}}{u_{2k+1} u_{2k+2}} v_{2k+1} x + \alpha \beta U_k \]

\[ = \frac{1}{(u)_{2k+2}} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (\alpha \beta)^{-j} x^{j+1} - jk \]

(4.14)

\[ \cdot \left[ (\alpha \beta)^{-k} \frac{v_{2k+1}}{u_{j+1}} - \alpha \beta \left( \frac{1}{u_{j+1}} + \frac{1}{u_{j+2}} \right) \right]. \]

5. APPLICATION TO FIBONACCI SUMMATIONS

We now consider the special case

(5.1) \[ \alpha + \beta = 1, \quad \alpha \beta = -1. \]

Then

(5.2) \[ u_n = F_n, \quad v = L_n, \]

the Fibonacci and Lucas numbers, respectively. Also \( U_k(x) \) becomes

(5.3) \[ \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{F_n F_{n+1} \cdots F_{n+2k}} \]

and in particular \( U_k \) becomes

(5.4) \[ \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{F_n F_{n+1} \cdots F_{n+2k}} = (-1)^k S_k. \]

Formula (4.10) reduces to
\[
\sum_{n=1}^{\infty} \frac{(-1)^{nk} x^{n+2k}}{F_n F_{n+1} \cdots F_{n+2k}} = \frac{1}{(F)_{2k}} \prod_{j=1}^{k} \left( x^2 + (-1)^j L_{2j-1} x - 1 \right) \cdot \sum_{n=1}^{\infty} \frac{x^n}{F_n}
\]

(5.5)

\[
- \left( \frac{1}{(F)_{2k}} \sum_{j=0}^{2k} (-1)^{\frac{j}{2}(j-1)-jk} \left\{ \frac{2k}{j} \right\} x^{2k-j} \sum_{n=1}^{\infty} \frac{x^n}{F_n} \right),
\]

where now

\[
\left\{ \frac{n}{j} \right\} = \frac{F_n F_{n-1} \cdots F_{n-j+1}}{F_F \cdots F_j}
\]

and

\[
(F)_{2k} = F_1 F_2 \cdots F_{2k}.
\]

In particular, for \( x = 1, -1 \), (5.5) reduces to

\[
\sum_{n=1}^{\infty} \frac{(-1)^{nk} x^{n+2k}}{F_n F_{n+1} \cdots F_{n+2k}} = \frac{(-1)^{\frac{1}{2}k(k+1)}}{(F)_{2k}} \prod_{j=1}^{k} L_{2j-1} \cdot \sum_{n=1}^{\infty} \frac{1}{F_n}
\]

(5.6)

\[
- \frac{1}{(F)_{2k}} \sum_{j=0}^{2k} (-1)^{\frac{j}{2}(j-1)-jk} \left\{ \frac{2k}{j} \right\} \sum_{n=1}^{\infty} \frac{1}{F_n},
\]

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n(k+1)}}{F_n F_{n+1} \cdots F_{n+2k}} = \frac{(-1)^{\frac{1}{2}k(k-1)}}{(F)_{2k}} \prod_{j=1}^{k} L_{2j-1} \cdot \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n}
\]

(5.7)

\[
- \frac{1}{(F)_{2k}} \sum_{j=0}^{2k} (-1)^{\frac{j}{2}(j+1)-jk} \left\{ \frac{2k}{j} \right\} \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n}.
\]

For example,
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\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} F_{n+2}} = -S + 3, \]

\[ \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}} = -\frac{2}{3} S + \frac{41}{18}, \]

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} \cdots F_{n+6}} = \frac{11}{60} S + \frac{17749}{23800}, \]

where

\[ S = \sum_{n=1}^{\infty} \frac{1}{F_n}. \]

We note also that (4.14) yields

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} \cdots F_{n+2k+2}} + \frac{(-1)^k}{F_{2k+1} F_{2k+2}} \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} \cdots F_{n+2k}} \]

\[ = \frac{1}{(F)^{2k+2}} \sum_{j=0}^{2k} (-1)^{\frac{j(j-1)}{2} - jk} \left( \sum_{j} \frac{(-1)^k}{F_{j+1}} + \frac{1}{F_{j+1}} + \frac{1}{F_{j+2}} \right). \]

For example,

\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} F_{n+2}} + \sum_{n=1}^{\infty} \frac{1}{F_n} = 3, \]

\[ \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} \cdots F_{n+4}} - \frac{2}{3} \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+1} F_{n+2}} = \frac{5}{18}, \]
in agreement with the special results obtained in [1].

It should be observed that the formulas of this section depend essentially on $\alpha \beta = -1$. Very similar results can be stated for $\alpha \beta = 1$. Thus, in particular we can obtain results like the above for such sums as

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n} F_{2n+2} \cdots F_{2n+4k}}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{nk}}{F_{3n} F_{3n+3} \cdots F_{3n+6k}}$$

6. SOME ADDITIONAL RESULTS

Returning to the general case, we shall now evaluate the sum

$$(6.1) \quad T_k = \sum_{n=1}^{\infty} \frac{(\alpha \beta)^{nk}}{u_n u_{n+1} \cdots u_{n+2k-1}} \quad (k = 1, 2, 3, \cdots) .$$

Multiplying (4.4) by $(\alpha \beta)^n / u_{n+2k+1}$, we get

$$\frac{(\alpha \beta)^{n(k+1)}}{u_n u_{n+1} \cdots u_{n+2k+1}} = \frac{1}{(a)^{2k}} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} \frac{(\alpha \beta)\frac{j(j+1)-jk+n}{u_{n+j+1} u_{n+2k+1}}}$$

so that
\[
\sum_{n=1}^{\infty} \frac{(\alpha \beta)^{n(k+1)}}{u_n u_{n+1} \cdots u_{n+2k+1}} = \frac{1}{u_{2k}} \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} (\alpha \beta)^{\frac{k}{2} j(j+1) - j(k+1)} \cdot \sum_{n=1}^{\infty} \frac{(\alpha \beta)^{n+j}}{u_{n+j} u_{n+2k+1}}.
\]  

(6.2)

Now consider the sum

\[
A_r = \sum_{n=1}^{\infty} \frac{(\alpha \beta)^n}{u_n u_{n+r}}.
\]  

(6.3)

Since

\[
u_n v_{n-1} - u_n u_{n+r-1} = (\alpha \beta)^n u_r,
\]

we have

\[
\frac{u_{n-1}}{u_n} - \frac{u_{n+r-1}}{u_{n+r}} = \frac{(\alpha \beta)^n u_r}{u_n u_{n+r}}.
\]

In this identity, take \( n = 1, 2, \cdots, N \) and sum. Then

\[
u_r \sum_{n=1}^{N} \frac{(\alpha \beta)^n}{u_n u_{n+r}} = \sum_{n=1}^{N} \frac{u_{n-1}}{u_n} - \sum_{n=1}^{N} \frac{u_{n+r-1}}{u_{n+r}}
\]  

(6.4)

\[
= \sum_{n=1}^{r} \frac{u_{n-1}}{u_n} - \sum_{n=1}^{r} \frac{u_{N+n-1}}{u_{N+n}}.
\]

Since we have assumed that

[Continued on page 510.]