COMBINATIONS AND THEIR DUALS

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In [3] this author gave derivations of certain results for restricted combinations by simple extensions of the first problem in Riordan [4, p. 14]. In these derivations k-combinations of the first n natural numbers were obtained by one-one correspondence with arrangements of plus signs and minus signs on a line. In what follows "dual" results are obtained by the symmetric interchange of the pluses and minuses.

For notation, terminology, and basic combinatorial results we follow Riordan [4]. By k-combinations will be meant k-combinations of the first n natural numbers.

To establish the correspondence, consider the arrangements of p pluses and q minuses on a line. If $p = k$ and $q = n - k$, each arrangement corresponds in a one-one way with a k-combination of the first n natural numbers as follows. Arrange the first n natural numbers on a line in their natural (rising) order; place a plus sign under each integer selected and a minus sign under each integer not selected.

It is well known that there are

$$\binom{p + q}{p}$$

arrangements of p pluses and q minuses on a line. With $p = k$ and $q = n - k$ we get the familiar

$$C(n,k) = \binom{n}{k}$$

k-combinations. The dual in this case gives nothing new since

$$C(n,k) = C(n,n - k) .$$

Starting with the first problem in Riordan [4, p. 14], with pluses and minuses interchanged, there are

$$\binom{q+1}{p} \tag{1}$$

arrangements of p pluses and q minuses on a line with no two pluses together [3]. With $p = k$ and $q = n - k$ we get Kaplansky's result [4, p. 198] that there are

$$\binom{n-k+1}{k} \tag{2}$$

k-combinations with no two consecutive integers in the same combination.

To get the dual in this case, interchange p and q in (1). Then with $p = k$ and $q = n - k$ we have that there are

$$\binom{k+1}{n-k} \tag{3}$$

k-combinations with no two consecutive integers omitted from the same combination $(n-1)/2 \le k \le n$.

In [3] we also rederived the circular case of Kaplansky's lemma [4, p. 198]. That is, there are

$$\frac{p+q}{q}\binom{q}{p} \tag{4}$$

arrangements of p pluses and q minuses on a circle with no two consecutive pluses, and

$$\frac{n}{n-k}\binom{n-k}{k}$$

circular k-combinations with no two consecutive integers, where n and 1 are taken to be consecutive. The dual in this case is that there are

$$\frac{n}{k}\binom{k}{n-k}$$

circular k-combinations with no two consecutive integers omitted, $n/2 \le k \le n$.

In rederiving (5) below, a result of Abramson and Moser [2], which generalizes (2), we got that there are

$$\binom{p-1}{r-1}\binom{q+1}{r}$$

arrangements of p pluses and q minuses on a line with exactly r blocks of consecutive pluses. With $p = k$ and $q = n - k$ there are

$$\binom{k-1}{r-1}\binom{n-k+1}{r} \tag{5}$$

k-combinations with exactly r blocks of consecutive integers. This reduces to (2) when $r = k$. The dual in this case is

$$\binom{n-k-1}{r-1}\binom{k+1}{r}$$

k-combinations with exactly r blocks of consecutive integers omitted. This reduces to (3) when $r = n - k$.

There are circular k-combinations corresponding to (5), see [2] or [3], and the appropriate dual.

Another generalization of (2) is that there are

$$\binom{q+p-bp+b}{p}$$

arrangements of p pluses and q minuses on a line with at least b minuses between any two pluses [3], and

$$\binom{n-bk+b}{k} \tag{6}$$

k-combinations such that if i occurs in a combination, none of $i+1$, $i+2$, $\cdots$, $i+b$ can [4, p. 222]. Here the dual is

$$\binom{n - b(n-k) + b}{n-k}$$

k-combinations such that if i is omitted, none of the $i+1$, $i+2$, $\cdots$, $i+b$ are, $b(n-1)/(b+1) \le k \le n$.

For the circular k-combinations corresponding to (6) see [3, (5b)] or [4, p. 222]. The dual follows readily from [3, (9b)].

Combining the restrictions in (5) and (6), we have

$$\binom{p-1}{r-1}\binom{q-(b-1)(r-1)+1}{r}$$

arrangements of p pluses and q minuses on a line with exactly r blocks of pluses, each block separated by at least b minuses. Thus there are

$$\binom{k-1}{r-1}\binom{n-k-(b-1)(r-1)+1}{r}$$

k-combinations with r blocks of consecutive integers with atlease b consecutive integers omitted between each block [3, (4b)]. The dual is

$$\binom{n-k-1}{r-1}\binom{k-(b-1)(r-1)+1}{r}$$

k-combinations with $r+1$ blocks of at least b consecutive integers in each, since there are only r gaps.

Clearly, additional results of the type we have considered above can be obtained from similar enumerations in the literature. Additional enumerations for which the duals are immediate appear in [3].

In closing, one enumeration and its dual should be mentioned. Expansion of the enumerating generating function

$$(1 + t + t^2 + \cdots + t^j)^{q+1}$$

gives

$$f(p,q;\ j+1) = \sum_{r=0}^{\left[\frac{p}{j+1}\right]} (-1)^r \binom{q+1}{r} \binom{q+p-r(j+1)}{q} ,$$

the number of arrangements of p pluses and q minuses on a line with at most j pluses between two minuses, before the first minus, and after the last. With $p = k$ and $q = n - k$ we get Abramson's [1]

$$A_{j+1}(n,k) = \sum_{r=0}^{\left[\frac{k}{j+1}\right]} (-1)^r \binom{n-k+1}{r} \binom{n-r(j+1)}{n-k} ,$$

the number of k-combinations with blocks of at most j consecutive integers. Its dual is

$$\sum_{r=0}^{\left[\frac{n-k}{j+1}\right]} (-1)^r \binom{k+1}{r} \binom{n-r(j+1)}{k} ,$$

the number of k-combinations with blocks of at most j consecutive integers omitted. See also (5).

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